Heat and Mass Transfer: Fundamentals & Applications
Fourth Edition
Yunus A. Cengel, Afshin J. Ghajar
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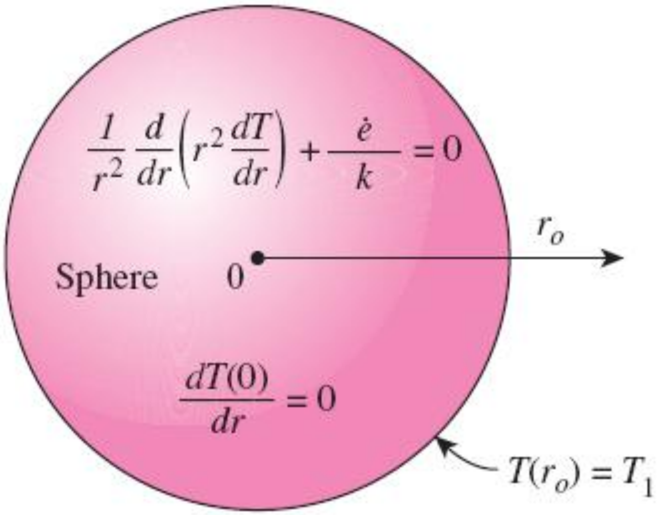
Chapter 5
NUMERICAL METHODS IN HEAT CONDUCTION

Objectives

- Understand the limitations of analytical solutions of conduction problems, and the need for computation-intensive numerical methods
- Express derivates as differences, and obtain finite difference formulations
- Solve steady one- or two-dimensional conduction problems numerically using the finite difference method
- Solve transient one- or two-dimensional conduction problems using the finite difference method

WHY NUMERICAL METHODS?



Solution:

$$T(r)=T_1+\frac{\dot{e}}{6k}(r_o^2-r^2)$$

$$\dot{Q}(r)=-kA\frac{dT}{dr}=\frac{4\pi r^3\dot{e}}{3}$$

FIGURE 5–1
The analytical solution of a problem requires solving the governing differential equation and applying the boundary conditions.

In Chapter 2, we solved various heat conduction problems in various geometries in a systematic but highly mathematical manner by

(1) deriving the governing differential equation by performing an energy balance on a differential volume element,

(2) expressing the boundary conditions in the proper mathematical form, and

(3) solving the differential equation and applying the boundary conditions to determine the integration constants.

1 Limitations

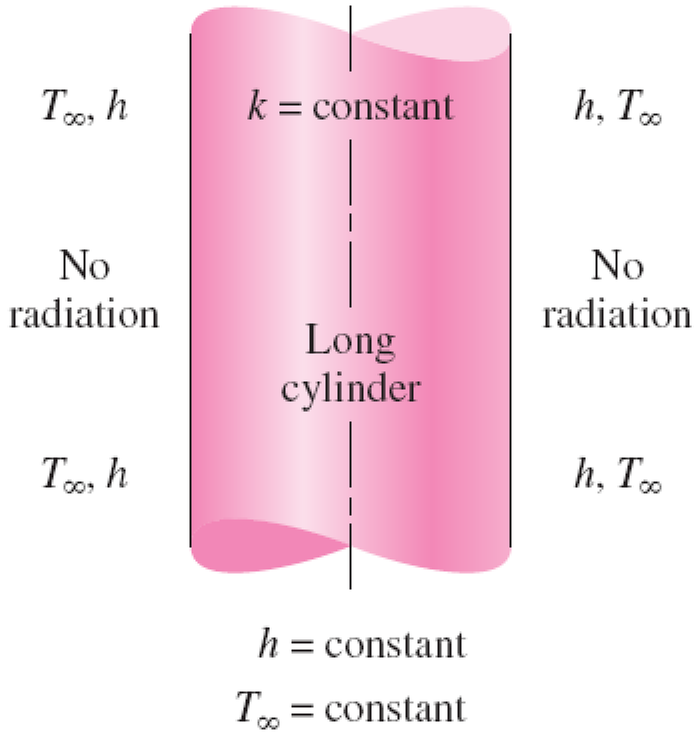


FIGURE 5–2
Analytical solution methods are limited to simplified problems in simple geometries.

Analytical solution methods are limited to *highly simplified problems* in *simple geometries*.

The geometry must be such that its entire surface can be described mathematically in a coordinate system by setting the variables equal to constants.

That is, it must fit into a coordinate system *perfectly* with nothing sticking out or in.

Even in simple geometries, heat transfer problems cannot be solved analytically if the *thermal conditions* are not sufficiently simple.

Analytical solutions are limited to problems that are simple or can be simplified with reasonable approximations.

2 Better Modeling

When attempting to get an analytical solution to a physical problem, there is always the tendency to *oversimplify* the problem to make the mathematical model sufficiently simple to warrant an analytical solution.

Therefore, it is common practice to ignore any effects that cause mathematical complications such as nonlinearities in the differential equation or the boundary conditions (*nonlinearities* such as temperature dependence of thermal conductivity and the radiation boundary conditions).

A mathematical model intended for a numerical solution is likely to represent the actual problem better.

The numerical solution of engineering problems has now become the norm rather than the exception even when analytical solutions are available.

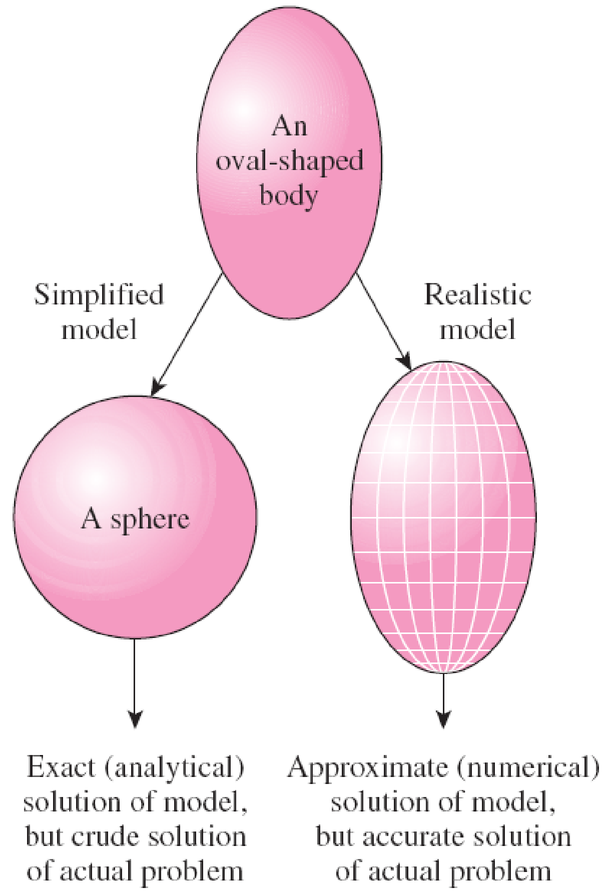


FIGURE 5–3
The approximate numerical solution of a real-world problem may be more accurate than the exact (analytical) solution of an oversimplified model of that problem.

3 Flexibility

Engineering problems often require extensive *parametric studies* to understand the influence of some variables on the solution in order to choose the right set of variables and to answer some “what-if” questions.

This is an *iterative process* that is extremely tedious and time-consuming if done by hand.

Computers and numerical methods are ideally suited for such calculations, and a wide range of related problems can be solved by minor modifications in the code or input variables.

Today it is almost unthinkable to perform any significant optimization studies in engineering without the power and flexibility of computers and numerical methods.

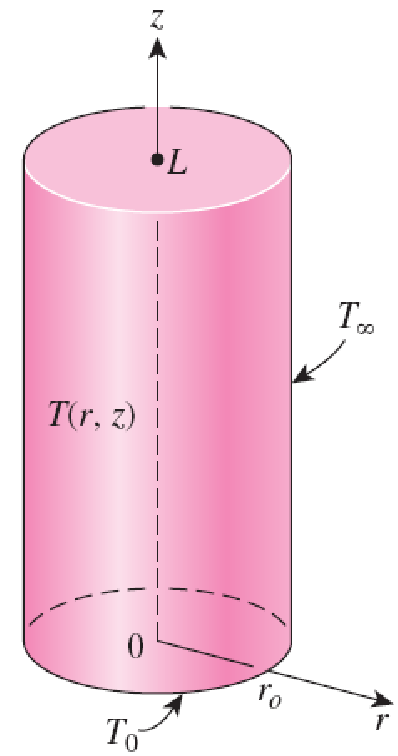
4 Complications

Some problems can be solved analytically, but the solution procedure is so complex and the resulting solution expressions so complicated that it is not worth all that effort.

With the exception of steady one-dimensional or transient lumped system problems, all heat conduction problems result in *partial* differential equations.

Solving such equations usually requires mathematical sophistication beyond that acquired at the undergraduate level, such as orthogonality, eigenvalues, Fourier and Laplace transforms, Bessel and Legendre functions, and infinite series.

In such cases, the evaluation of the solution, which often involves double or triple summations of infinite series at a specified point, is a challenge in itself.



Analytical solution:

$$\frac{T(r, z) - T_\infty}{T_0 - T_\infty} = \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n r_o)} \frac{\sinh \lambda_n (L - z)}{\sinh (\lambda_n L)}$$

where λ_n's are roots of $J_0(\lambda_n r_o) = 0$

FIGURE 5–4
Some analytical solutions are very complex and difficult to use.

5 Human Nature

FIGURE 5–5
The ready availability of high-powered computers with sophisticated software packages has made numerical solution the norm rather than the exception.

Analytical solutions are necessary because insight to the *physical phenomena* and *engineering wisdom* is gained primarily through analysis.

The "feel" that engineers develop during the analysis of simple but fundamental problems serves as an invaluable tool when interpreting a huge pile of results obtained from a computer when solving a complex problem.

A simple analysis by hand for a limiting case can be used to check if the results are in the proper range.

In this chapter, you will learn how to *formulate* and *solve* heat transfer problems numerically using one or more approaches.

FINITE DIFFERENCE FORMULATION OF DIFFERENTIAL EQUATIONS

The numerical methods for solving differential equations are based on replacing the *differential equations* by *algebraic equations.*

In the case of the popular **finite difference** method, this is done by replacing the *derivatives* by *differences.*

Below we demonstrate this with both first- and second-order derivatives.

AN EXAMPLE

$$A = A_0(1 + i)^n = (\$100)(1 + 0.09)^2 = \$118.81$$

$$dA/dt = iA \qquad A = A_0 \exp(it)$$

$$A = (\$100)\exp(0.18 \times 1) = \$119.72$$

Reasonably accurate results can be obtained by replacing differential quantities by sufficiently small differences

TABLE 5–1

Year-end balance of a $100 account earning interest at an annual rate of 18 percent for various compounding periods

Compounding Period	Number of Periods, n	Year-End Balance
1 year	1	$118.00
6 months	2	118.81
1 month	12	119.56
1 week	52	119.68
1 day	365	119.72
1 hour	8760	119.72
1 minute	525,600	119.72
1 second	31,536,000	119.72
Instantaneous	∞	119.72

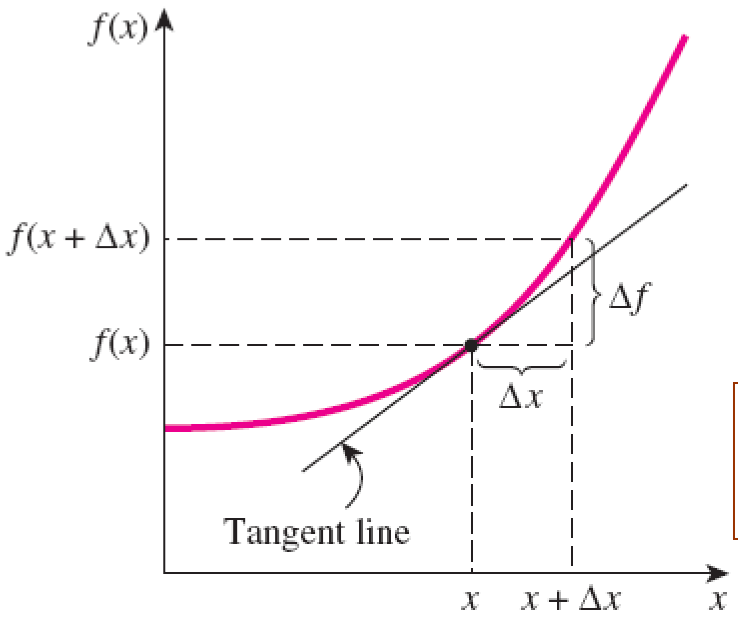


FIGURE 5–6
The derivative of a function at a point represents the slope of the function at that point.

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{df(x)}{dx} \cong \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

finite difference form of the first derivative

$$f(x + \Delta x) = f(x) + \Delta x \frac{df(x)}{dx} + \frac{1}{2} \Delta x^2 \frac{d^2 f(x)}{dx^2} + \cdots$$

Taylor series expansion of the function *f* about the point *x*,

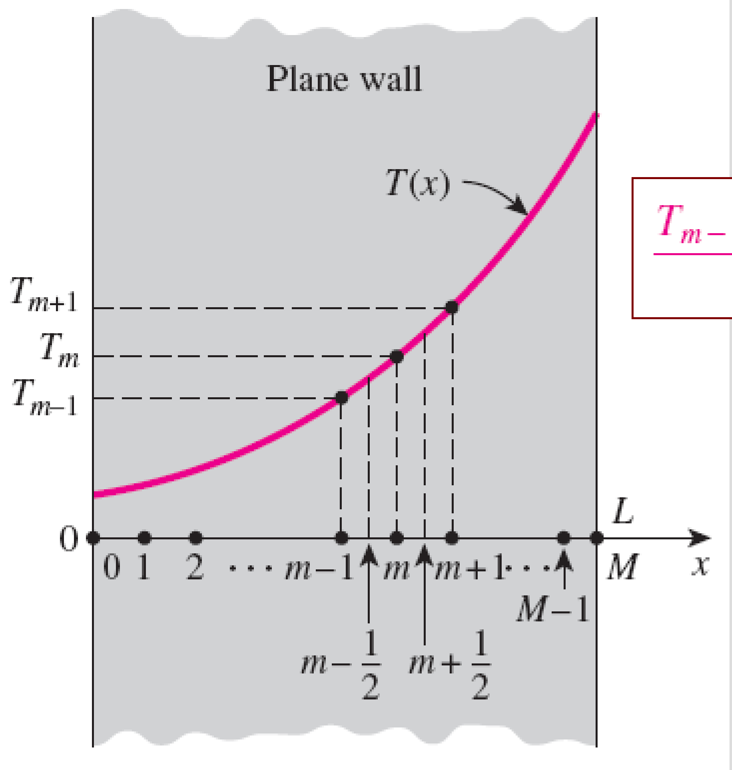
The smaller the Δx, the smaller the error, and thus the more accurate the approximation.

Consider steady one-dimensional heat conduction in a plane wall of thickness *L* with heat generation.

$$\left.\frac{dT}{dx}\right|_{m-\frac{1}{2}} \cong \frac{T_m - T_{m-1}}{\Delta x} \quad \text{and} \quad \left.\frac{dT}{dx}\right|_{m+\frac{1}{2}} \cong \frac{T_{m+1} - T_m}{\Delta x}$$

Finite difference representation of the *second derivative* at a general internal node *m.*

$$\left.\frac{d^2T}{dx^2}\right|_m \cong \frac{\left.\frac{dT}{dx}\right|_{m+\frac{1}{2}} - \left.\frac{dT}{dx}\right|_{m-\frac{1}{2}}}{\Delta x} = \frac{\frac{T_{m+1} - T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x} = \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2}$$



$$\frac{d^2T}{dx^2} + \frac{\dot{e}}{k} = 0$$

$$\boxed{\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0, \qquad m = 1, 2, 3, \ldots, M-1}$$

$$T_m = ½(T_{m-1} + T_{m+1})$$ no heat generation

Schematic of the nodes and the nodal temperatures used in the development of the finite difference formulation of heat transfer in a plane wall.

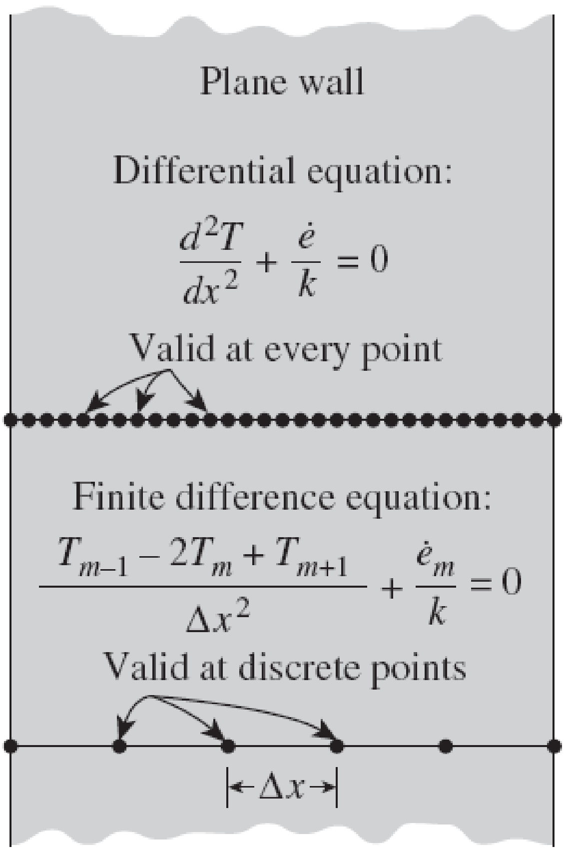


FIGURE 5–8

The differential equation is valid at every point of a medium, whereas the finite difference equation is valid at discrete points (the nodes) only.

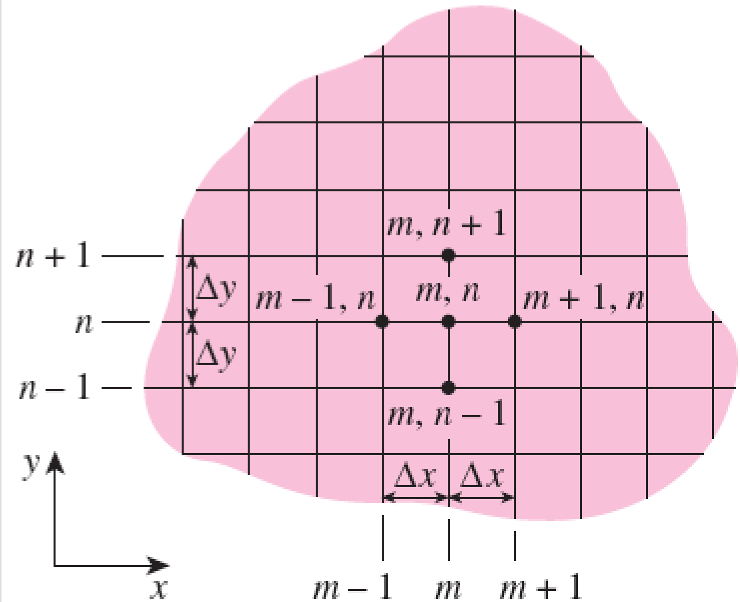


FIGURE 5–9

Finite difference mesh for two-dimensional conduction in rectangular coordinates.

Finite difference formulation for **steady two-dimensional heat conduction** in a region with heat generation and constant thermal conductivity in rectangular coordinates

$$\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{\Delta x^2} + \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{\Delta y^2} + \frac{\dot{e}_{m,n}}{k} = 0$$

ONE-DIMENSIONAL STEADY HEAT CONDUCTION

In this section we develop the finite difference formulation of heat conduction in a plane wall using the energy balance approach and discuss how to solve the resulting equations.

The **energy balance method** is based on *subdividing* the medium into a sufficient number of volume elements and then applying an *energy balance* on each element.

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at the left}\\ \text{surface}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at the right}\\ \text{surface}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$

$$\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, right}} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

$$\Delta E_{\text{element}} = 0$$

$$\dot{E}_{\text{gen, element}} = \dot{e}_m V_{\text{element}} = \dot{e}_m A \Delta x \qquad \dot{Q}_{\text{cond}} = kA\frac{\Delta T}{L}$$

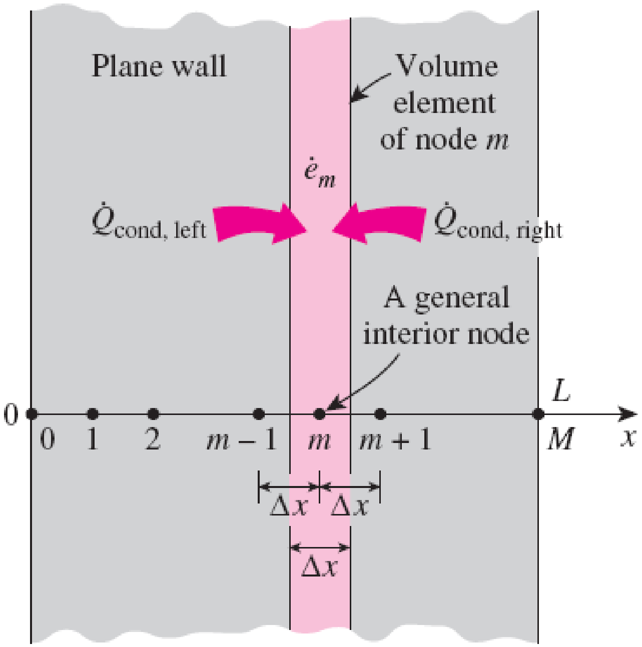


FIGURE 5–10

The nodal points and volume elements for the finite difference formulation of one-dimensional conduction in a plane wall.

$$\dot{Q}_{\text{cond, left}} = kA\frac{T_{m-1} - T_m}{\Delta x}$$

$$\dot{Q}_{\text{cond, right}} = kA\frac{T_{m+1} - T_m}{\Delta x}$$

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + \dot{e}_m A\Delta x = 0$$

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0,$$

$$m = 1, 2, 3, \ldots, M - 1$$

This equation is applicable to each of the M - 1 interior nodes, and its application gives M - 1 equations for the determination of temperatures at M + 1 nodes.

The two additional equations needed to solve for the M + 1 unknown nodal temperatures are obtained by applying the energy balance on the two elements at the boundaries (unless, of course, the boundary temperatures are specified).

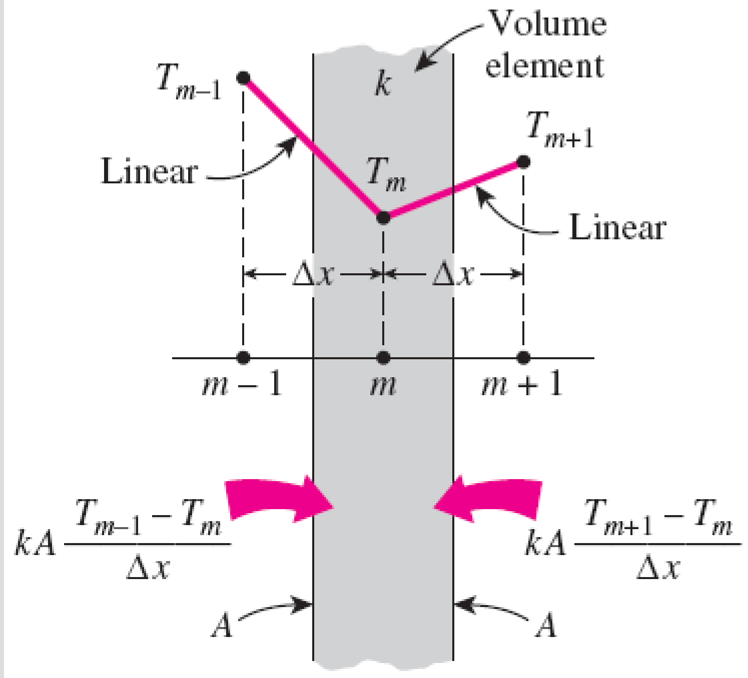
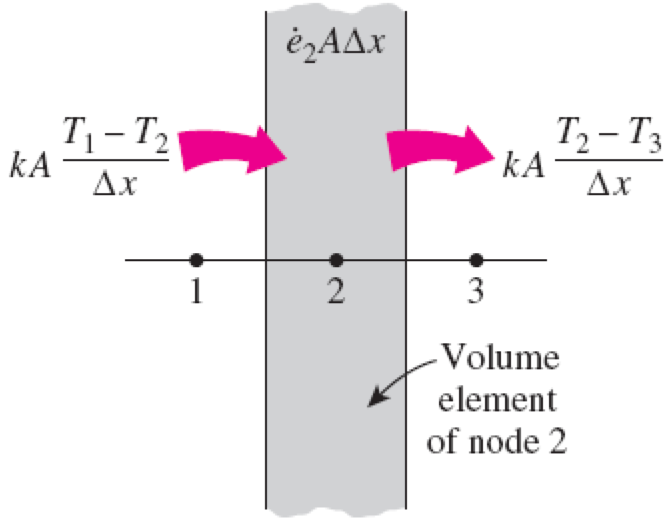


FIGURE 5–11

In finite difference formulation, the temperature is assumed to vary linearly between the nodes.

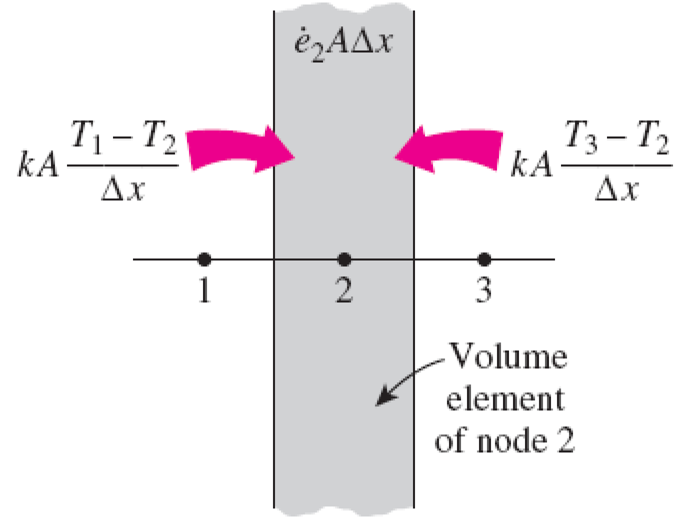


$$kA\frac{T_1 - T_2}{\Delta x} - kA\frac{T_2 - T_3}{\Delta x} + \dot{e}_2 A \Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{e}_2 A \Delta x^2 / k = 0$$

(*a*) Assuming heat transfer to be out of the volume element at the right surface.



$$kA\frac{T_1 - T_2}{\Delta x} + kA\frac{T_3 - T_2}{\Delta x} + \dot{e}_2 A \Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{e}_2 A \Delta x^2 / k = 0$$

(*b*) Assuming heat transfer to be into the volume element at all surfaces.

FIGURE 5–12

The assumed direction of heat transfer at surfaces of a volume element has no effect on the finite difference formulation.

Boundary Conditions

Boundary conditions most commonly encountered in practice are the *specified temperature, specified heat flux, convection,* and *radiation* boundary conditions, and here we develop the finite difference formulations for them for the case of steady one-dimensional heat conduction in a plane wall of thickness L as an example.

The node number at the left surface at $x = 0$ is 0, and at the right surface at $x = L$ it is M. Note that the width of the volume element for either boundary node is $\Delta x/2$.

Specified temperature boundary condition

$$T(0) = T_0 = \text{Specified value}$$
$$T(L) = T_M = \text{Specified value}$$

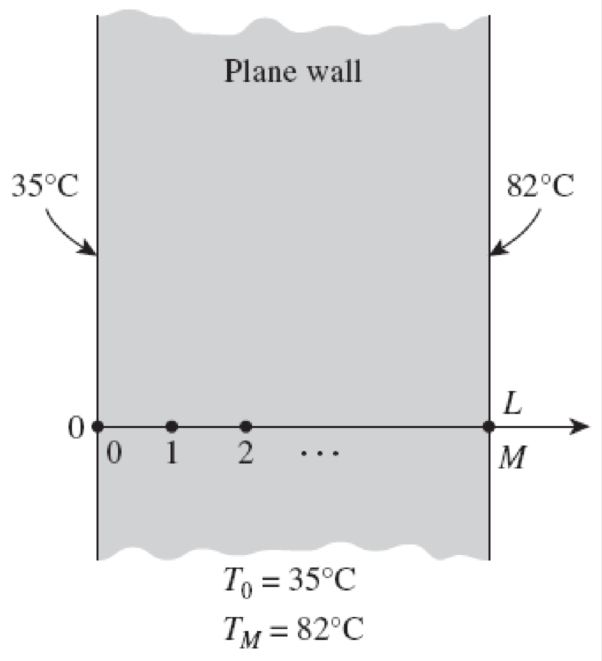


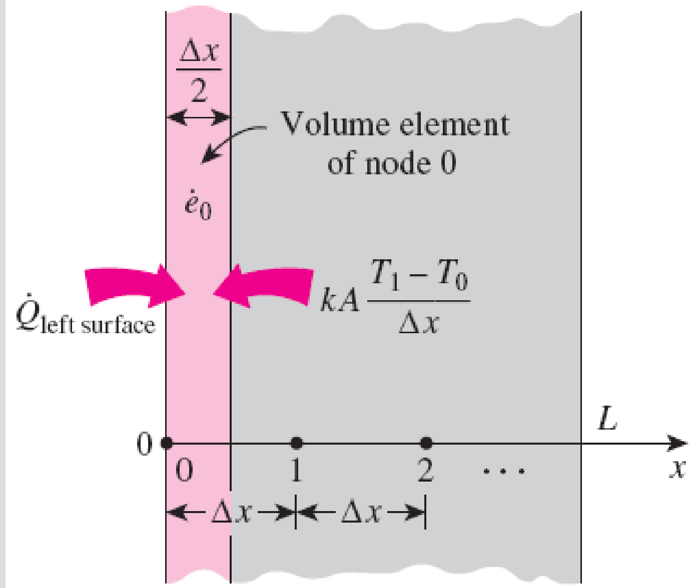
FIGURE 5–13

Finite difference formulation of specified temperature boundary conditions on both surfaces of a plane wall.

When other boundary conditions such as the *specified heat flux, convection, radiation,* or *combined convection and radiation* conditions are specified at a boundary, the finite difference equation for the node at that boundary is obtained by writing an *energy balance* on the volume element at that boundary.

$$\sum_{\text{All sides}} \dot{Q} + \dot{E}_{\text{gen, element}} = 0$$

$$\dot{Q}_{\text{left surface}} + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$



$$\dot{Q}_{\text{left surface}} + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0 A\frac{\Delta x}{2} = 0$$

FIGURE 5–14

Schematic for the finite difference formulation of the left boundary node of a plane wall.

The finite difference form of various boundary conditions at the left boundary:

1. Specified Heat Flux Boundary Condition

$$\dot{q}_0 A + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$

Special case: **Insulated Boundary** ($\dot{q}_0 = 0$)

$$kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$

2. Convection Boundary Condition

$$hA(T_\infty - T_0) + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$

3. Radiation Boundary Condition

$$\varepsilon\sigma A(T_{\text{surr}}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$

4. Combined Convection and Radiation Boundary Condition
(Fig. 5–15)

$$hA(T_\infty - T_0) + \varepsilon\sigma A(T_{\text{surr}}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$

or

$$h_{\text{combined}}\,A(T_\infty - T_0) + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$

5. Combined Convection, Radiation, and Heat Flux Boundary Condition

$$\dot{q}_0A + hA(T_\infty - T_0) + \varepsilon\sigma A(T_{\text{surr}}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$

6. Interface Boundary Condition Two different solid media A and B are assumed to be in perfect contact, and thus at the same temperature at the interface at node m (Fig. 5–16). Subscripts A and B indicate properties of media A *and* B, *respectively.*

$$k_A A \frac{T_{m-1} - T_m}{\Delta x} + k_B A \frac{T_{m+1} - T_m}{\Delta x} + \dot{e}_{A,m}(A\Delta x/2) + \dot{e}_{B,m}(A\Delta x/2) = 0 \qquad \textbf{(5–29)}$$

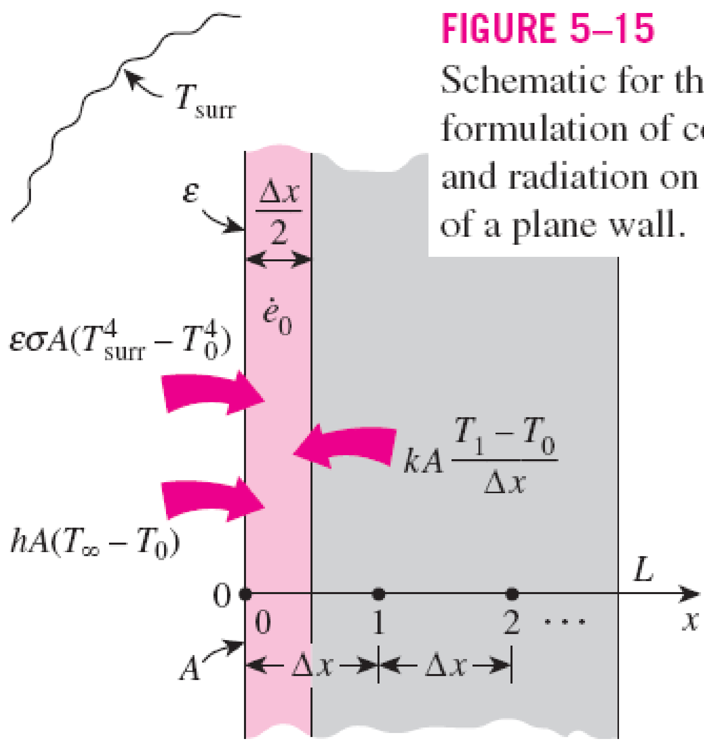
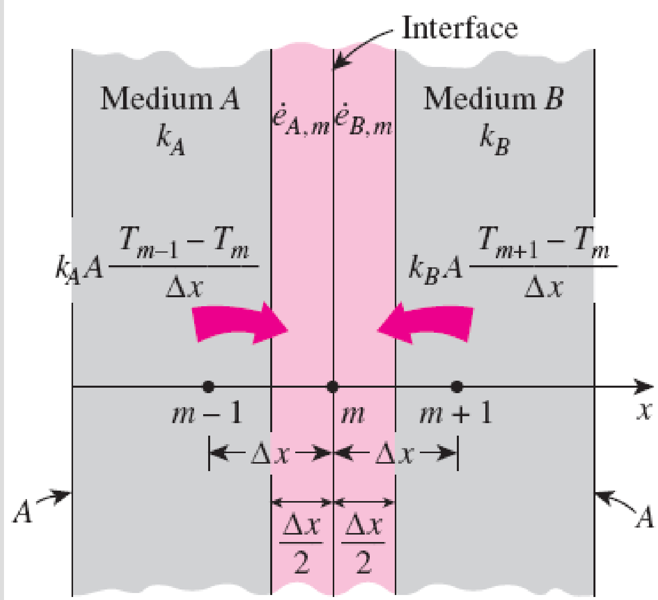


FIGURE 5–15
Schematic for the finite difference formulation of combined convection and radiation on the left boundary of a plane wall.

$$hA(T_\infty - T_0) + \varepsilon\sigma A(T_{surr}^4 - T_0^4)$$
$$+ kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0 A\frac{\Delta x}{2} = 0$$

Schematic for the finite difference formulation of the interface boundary condition for two mediums A and B that are in perfect thermal contact.



$$k_A A\frac{T_{m-1} - T_m}{\Delta x} + k_B A\frac{T_{m+1} - T_m}{\Delta x}$$
$$+ \dot{e}_{A,m}A\frac{\Delta x}{2} + \dot{e}_{B,m}A\frac{\Delta x}{2} = 0$$

Treating Insulated Boundary Nodes as Interior Nodes: The Mirror Image Concept

$$\frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \quad \rightarrow \quad \frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{\dot{e}_0}{k} = 0$$

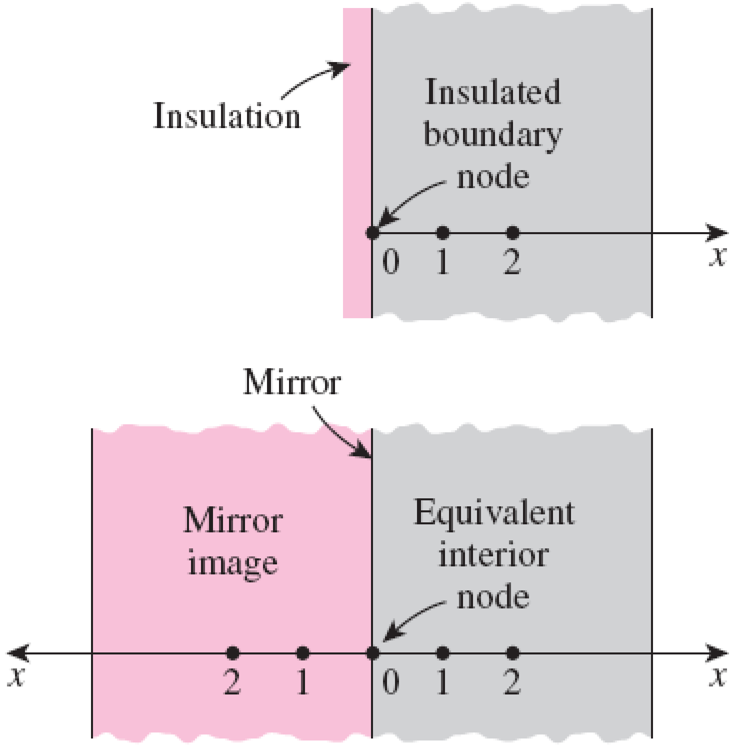


FIGURE 5–17

A node on an insulated boundary can be treated as an interior node by replacing the insulation by a mirror.

The mirror image approach can also be used for problems that possess thermal symmetry by replacing the plane of symmetry by a mirror.

Alternately, we can replace the plane of symmetry by insulation and consider only half of the medium in the solution.

The solution in the other half of the medium is simply the mirror image of the solution obtained.

EXAMPLE 5–1 Steady Heat Conduction in a Large Uranium Plate

Consider a large uranium plate of thickness $L = 4$ cm and thermal conductivity $k = 28$ W/m · °C in which heat is generated uniformly at a constant rate of $\dot{g} = 5 \times 10^6$ W/m³. One side of the plate is maintained at 0°C by iced water while the other side is subjected to convection to an environment at $T_\infty = 30$°C with a heat transfer coefficient of $h = 45$ W/m² · °C, as shown in Figure 5–18. Considering a total of three equally spaced nodes in the medium, two at the boundaries and one at the middle, estimate the exposed surface temperature of the plate under steady conditions using the finite difference approach.

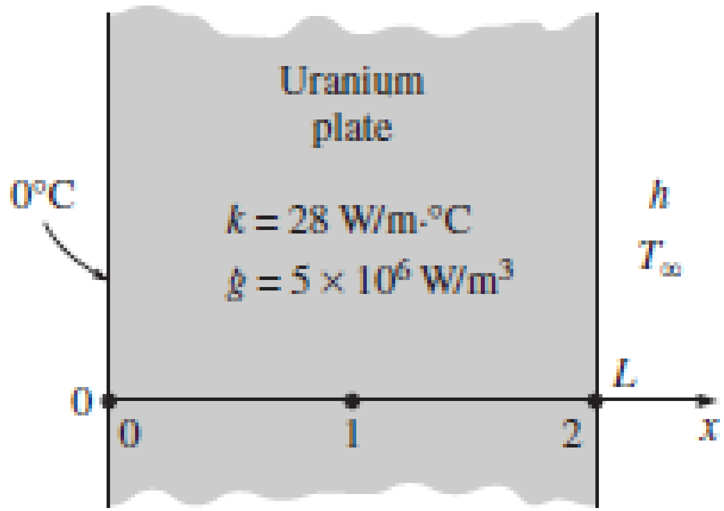


FIGURE 5–18
Schematic for Example 5–1.

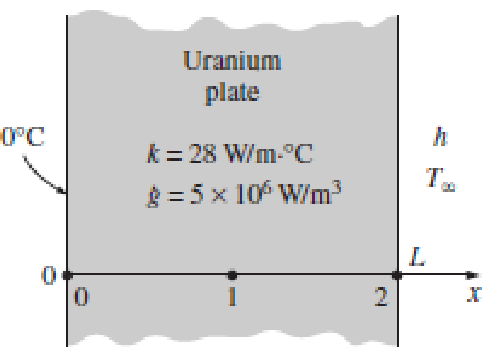


FIGURE 5–18

SOLUTION A uranium plate is subjected to specified temperature on one side and convection on the other. The unknown surface temperature of the plate is to be determined numerically using three equally spaced nodes.

Assumptions **1** Heat transfer through the wall is steady since there is no indication of any change with time. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Thermal conductivity is constant. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity is given to be $k = 28$ W/m · °C.

Analysis The number of nodes is specified to be $M = 3$, and they are chosen to be at the two surfaces of the plate and the midpoint, as shown in the figure. Then the nodal spacing Δx becomes

$$\Delta x = \frac{L}{M-1} = \frac{0.04 \text{ m}}{3-1} = 0.02 \text{ m}$$

We number the nodes 0, 1, and 2. The temperature at node 0 is given to be $T_0 = 0°\text{C}$, and the temperatures at nodes 1 and 2 are to be determined. This problem involves only two unknown nodal temperatures, and thus we need to have only two equations to determine them uniquely. These equations are obtained by applying the finite difference method to nodes 1 and 2.

Node 1 is an interior node, and the finite difference formulation at that node is obtained directly from Eq. 5–18 by setting $m = 1$:

$$\frac{T_0 - 2T_1 + T_2}{\Delta x^2} + \frac{\dot{g}_1}{k} = 0 \;\rightarrow\; \frac{0 - 2T_1 + T_2}{\Delta x^2} + \frac{\dot{g}_1}{k} = 0 \;\rightarrow\; 2T_1 - T_2 = \frac{\dot{g}_1 \Delta x^2}{k} \tag{1}$$

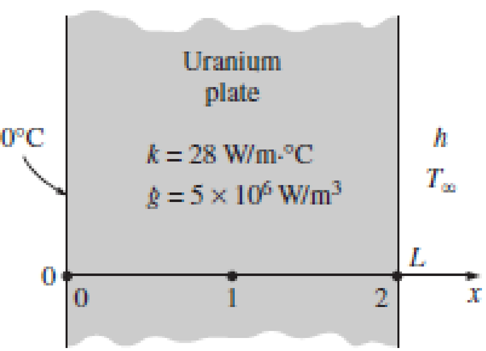


FIGURE 5–18

Node 2 is a boundary node subjected to convection, and the finite difference formulation at that node is obtained by writing an energy balance on the volume element of thickness $\Delta x/2$ at that boundary by assuming heat transfer to be into the medium at all sides:

$$hA(T_\infty - T_2) + kA\frac{T_1 - T_2}{\Delta x} + \dot{g}_2(A\Delta x/2) = 0$$

Canceling the heat transfer area A and rearranging give

$$T_1 - \left(1 + \frac{h\Delta x}{k}\right)T_2 = -\frac{h\Delta x}{k}T_\infty - \frac{\dot{g}_2\Delta x^2}{2k} \qquad \textbf{(2)}$$

Equations (1) and (2) form a system of two equations in two unknowns T_1 and T_2. Substituting the given quantities and simplifying gives

$$2T_1 - T_2 = 71.43 \qquad \text{(in °C)}$$
$$T_1 - 1.032T_2 = -36.68 \qquad \text{(in °C)}$$

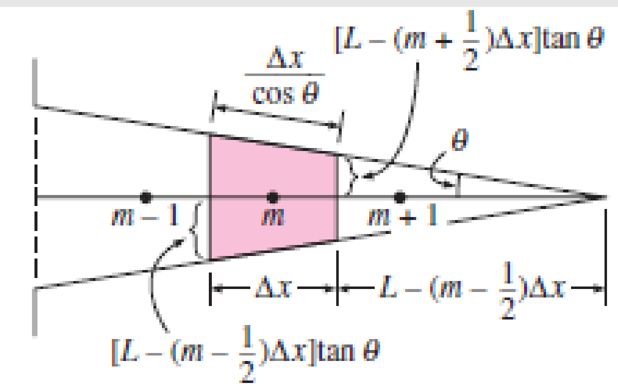
This is a system of two algebraic equations in two unknowns and can be solved easily by the elimination method. Solving the first equation for T_1 and substituting into the second equation result in an equation in T_2 whose solution is

$$T_2 = 136.1\text{°C}$$

This is the temperature of the surface exposed to convection, which is the desired result. Substitution of this result into the first equation gives $T_1 =$ 103.8°C, which is the temperature at the middle of the plate.

EXAMPLE 5–2 **Heat Transfer from Triangular Fins**

Consider an aluminum alloy fin (k = 180 W/m · °C) of triangular cross section with length L = 5 cm, base thickness b = 1 cm, and very large width w in the direction normal to the plane of paper, as shown in Figure 5–20. The base of the fin is maintained at a temperature of T_0 = 200°C. The fin is losing heat to the surrounding medium at T_∞ = 25°C with a heat transfer coefficient of h = 15 W/m² · °C. Using the finite difference method with six equally spaced nodes along the fin in the x-direction, determine (*a*) the temperatures at the nodes, (*b*) the rate of heat transfer from the fin for w = 1 m, and (*c*) the fin efficiency.



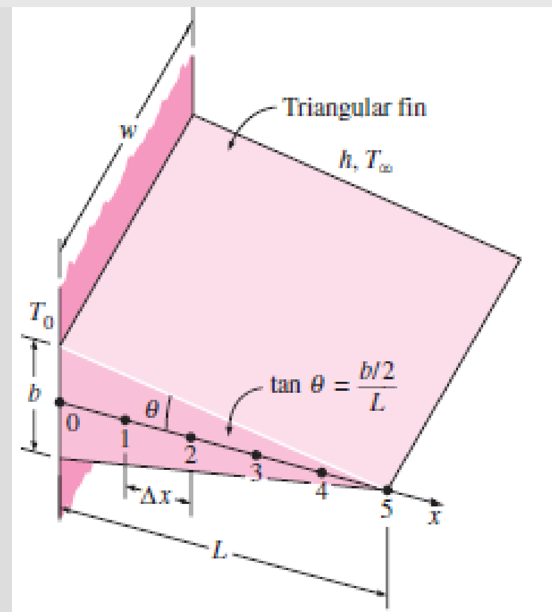


FIGURE 5–20
Schematic for Example 5–2 and the volume element of a general interior node of the fin.

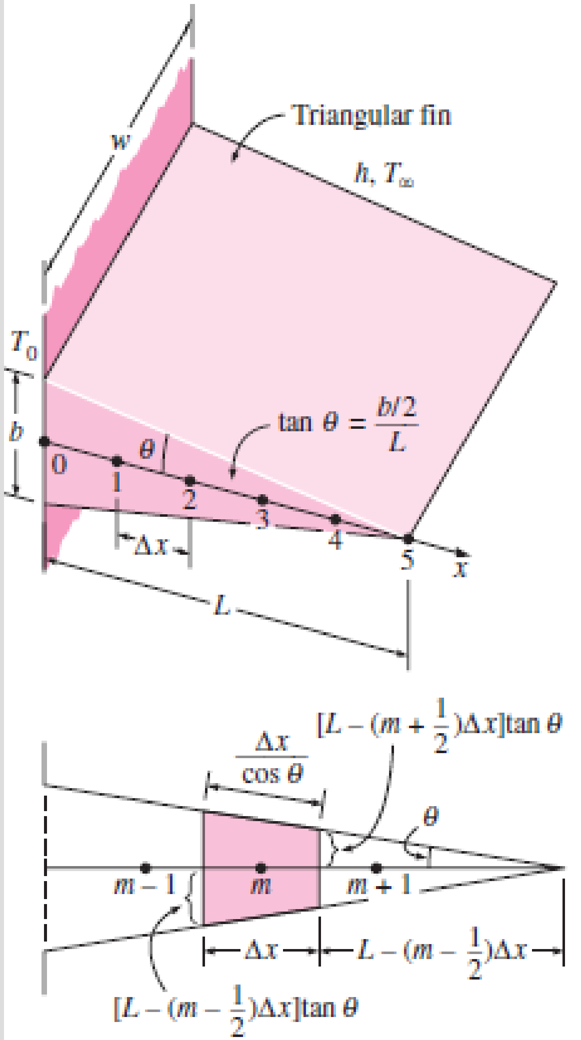


FIGURE 5–20

SOLUTION A long triangular fin attached to a surface is considered. The nodal temperatures, the rate of heat transfer, and the fin efficiency are to be determined numerically using six equally spaced nodes.

Assumptions **1** Heat transfer is steady since there is no indication of any change with time. **2** The temperature along the fin varies in the *x* direction only. **3** Thermal conductivity is constant. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity is given to be $k = 180$ W/m · °C.

Analysis (*a*) The number of nodes in the fin is specified to be $M = 6$, and their location is as shown in the figure. Then the nodal spacing Δx becomes

$$\Delta x = \frac{L}{M-1} = \frac{0.05\text{ m}}{6-1} = 0.01\text{ m}$$

The temperature at node 0 is given to be $T_0 = 200°\text{C}$, and the temperatures at the remaining five nodes are to be determined. Therefore, we need to have five equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and the finite difference formulation for a general interior node m is obtained by applying an energy balance on the volume element of this node. Noting that heat transfer is steady and there is no heat generation in the fin and assuming heat transfer to be into the medium at all sides, the energy balance can be expressed as

$$\sum_{\text{all sides}} \dot{Q} = 0 \quad \rightarrow \quad kA_{\text{left}} \frac{T_{m-1} - T_m}{\Delta x} + kA_{\text{right}} \frac{T_{m+1} - T_m}{\Delta x} + hA_{\text{conv}}(T_\infty - T_m) = 0$$

Note that heat transfer areas are different for each node in this case, and using geometrical relations, they can be expressed as

$$\begin{aligned}
A_{\text{left}} &= (\text{Height} \times \text{Width})_{@m-\frac{1}{2}} = 2w[L - (m - 1/2)\Delta x]\tan\theta \\
A_{\text{right}} &= (\text{Height} \times \text{Width})_{@m+\frac{1}{2}} = 2w[L - (m + 1/2)\Delta x]\tan\theta \\
A_{\text{conv}} &= 2 \times \text{Length} \times \text{Width} = 2w(\Delta x/\cos\theta)
\end{aligned}$$

Substituting,

$$2kw[L - (m - \tfrac{1}{2})\Delta x]\tan\theta\, \frac{T_{m-1} - T_m}{\Delta x}$$

$$+ 2kw[L - (m + \tfrac{1}{2})\Delta x]\tan\theta\, \frac{T_{m+1} - T_m}{\Delta x} + h\,\frac{2w\Delta x}{\cos\theta}(T_\infty - T_m) = 0$$

Dividing each term by $2kwL \tan\theta/\Delta x$ gives

$$\left[1-\left(m-\tfrac{1}{2}\right)\frac{\Delta x}{L}\right](T_{m-1}-T_m)+\left[1-\left(m+\tfrac{1}{2}\right)\frac{\Delta x}{L}\right](T_{m+1}-T_m)$$
$$+\frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty-T_m)=0$$

Note that

$$\tan\theta=\frac{b/2}{L}=\frac{0.5\text{ cm}}{5\text{ cm}}=0.1 \quad\rightarrow\quad \theta=\tan^{-1}0.1=5.71°$$

Also, $\sin 5.71° = 0.0995$. Then the substitution of known quantities gives

$$(5.5-m)T_{m-1}-(10.00838-2m)T_m+(4.5-m)T_{m+1}=-0.209$$

Now substituting 1, 2, 3, and 4 for m results in these finite difference equations for the interior nodes:

$$m=1:\quad -8.00838T_1+3.5T_2=-900.209 \qquad (1)$$
$$m=2:\quad 3.5T_1-6.00838T_2+2.5T_3=-0.209 \qquad (2)$$
$$m=3:\quad 2.5T_2-4.00838T_3+1.5T_4=-0.209 \qquad (3)$$
$$m=4:\quad 1.5T_3-2.00838T_4+0.5T_5=-0.209 \qquad (4)$$

The finite difference equation for the boundary node 5 is obtained by writing an energy balance on the volume element of length $\Delta x/2$ at that boundary, again by assuming heat transfer to be into the medium at all sides (Fig. 5–21):

$$kA_{\text{left}}\frac{T_4-T_5}{\Delta x}+hA_{\text{conv}}(T_\infty-T_5)=0$$

where

$$A_{\text{left}}=2w\frac{\Delta x}{2}\tan\theta \qquad \text{and} \qquad A_{\text{conv}}=2w\frac{\Delta x/2}{\cos\theta}$$

Canceling w in all terms and substituting the known quantities gives

$$T_4-1.00838T_5=-0.209 \qquad (5)$$

Equations (1) through (5) form a linear system of five algebraic equations in five unknowns. Solving them simultaneously using an equation solver gives

$$T_1=198.6°\text{C},\quad T_2=197.1°\text{C},\quad T_3=195.7°\text{C},$$
$$T_4=194.3°\text{C},\quad T_5=192.9°\text{C}$$

which is the desired solution for the nodal temperatures.

(*b*) The total rate of heat transfer from the fin is simply the sum of the heat transfer from each volume element to the ambient, and for $w = 1$ m it is determined from

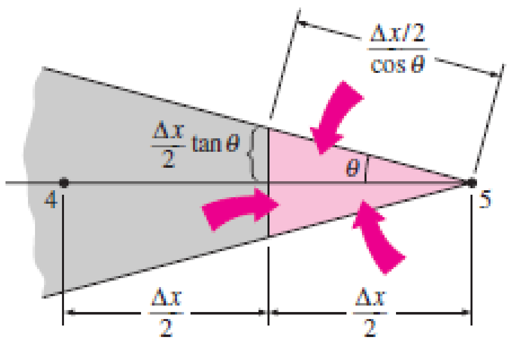


FIGURE 5–21
Schematic of the volume element of node 5 at the tip of a triangular fin.

$$\dot{Q}_{\text{fin}} = \sum_{m=0}^{5} \dot{Q}_{\text{element}, m} = \sum_{m=0}^{5} hA_{\text{conv}, m}(T_m - T_\infty)$$

Noting that the heat transfer surface area is $w\Delta x/\cos\theta$ for the boundary nodes 0 and 5, and twice as large for the interior nodes 1, 2, 3, and 4, we have

$$\begin{aligned}
\dot{Q}_{\text{fin}} &= h\frac{w\Delta x}{\cos\theta}[(T_0 - T_\infty) + 2(T_1 - T_\infty) + 2(T_2 - T_\infty) + 2(T_3 - T_\infty) \\
&\quad + 2(T_4 - T_\infty) + (T_5 - T_\infty)] \\
&= h\frac{w\Delta x}{\cos\theta}[T_0 + 2(T_1 + T_2 + T_3 + T_4) + T_5 - 10T_\infty] \\
&= (15\ \text{W/m}^2\cdot{}^\circ\text{C})\frac{(1\ \text{m})(0.01\ \text{m})}{\cos 5.71^\circ}[200 + 2 \times 785.7 + 192.9 - 10 \times 25] \\
&= \mathbf{258.4\ W}
\end{aligned}$$

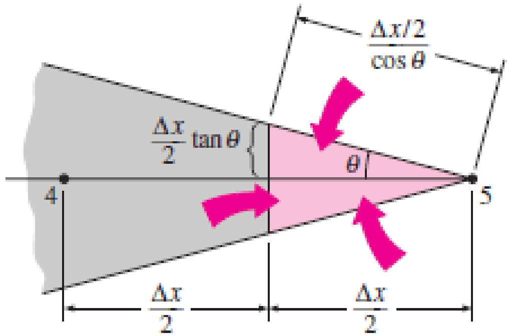


FIGURE 5–21
Schematic of the volume element of node 5 at the tip of a triangular fin.

(*c*) If the entire fin were at the base temperature of $T_0 = 200°\text{C}$, the total rate of heat transfer from the fin for $w = 1$ m would be

$$\begin{aligned}
\dot{Q}_{\max} &= hA_{\text{fin, total}}(T_0 - T_\infty) = h(2wL/\cos\theta)(T_0 - T_\infty) \\
&= (15\ \text{W/m}^2\cdot{}^\circ\text{C})[2(1\ \text{m})(0.05\ \text{m})/\cos 5.71^\circ](200 - 25)^\circ\text{C} \\
&= 263.8\ \text{W}
\end{aligned}$$

Then the fin efficiency is determined from

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\max}} = \frac{258.4\ \text{W}}{263.8\ \text{W}} = \mathbf{0.98}$$

which is less than 1, as expected. We could also determine the fin efficiency in this case from the proper fin efficiency curve in Chapter 3, which is based on the analytical solution. We would read 0.98 for the fin efficiency, which is identical to the value determined above numerically.

The finite difference formulation of steady heat conduction problems usually results in a system of *N* algebraic equations in *N* unknown nodal temperatures that need to be solved simultaneously.

There are numerous systematic approaches available in the literature, and they are broadly classified as **direct** and **iterative** methods.

The direct methods are based on a fixed number of well-defined steps that result in the solution in a systematic manner.

The iterative methods are based on an initial guess for the solution that is refined by iteration until a specified convergence criterion is satisfied.

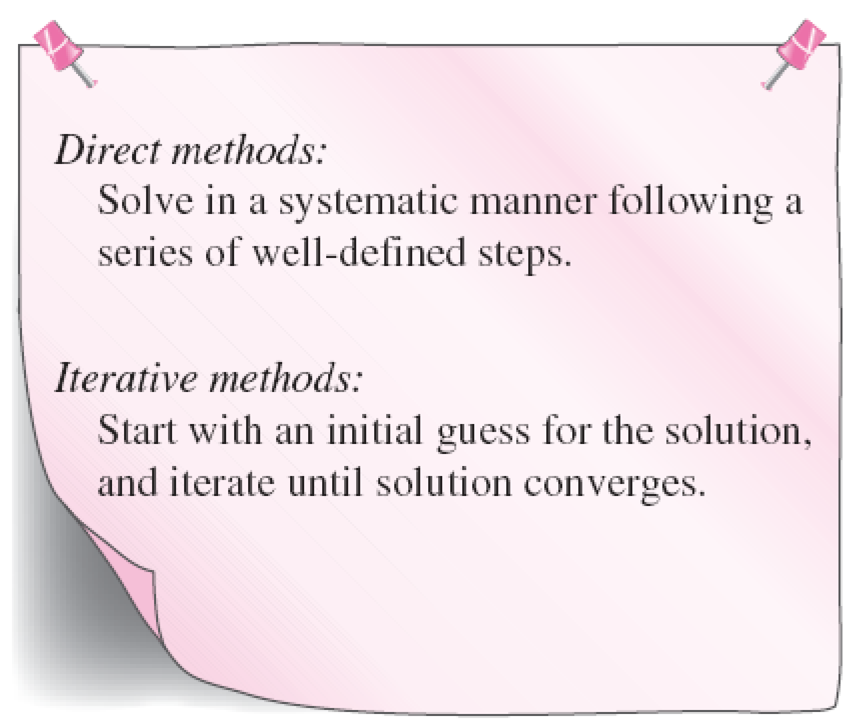


FIGURE 5–22

Two general categories of solution methods for solving systems of algebraic equations.

One of the simplest iterative methods is the *Gauss-Seidel* iteration.

TABLE 5–2

Application of the *Gauss-Seidel* iterative method to the finite difference equations of Example 5–2.

Finite difference equations in explicit form

$T_1 = 0.4371T_2 + 112.4137$

$T_2 = 0.5826T_1 + 0.4161T_3 + 0.0348$

$T_3 = 0.6238T_2 + 0.3743T_4 + 0.0521$

$T_4 = 0.7470T_3 + 0.2490T_5 + 0.1041$

$T_5 = 0.9921T_4 + 0.2073$

Iteration	Nodal Temperature, °C				
	T_1	T_2	T_3	T_4	T_5
Initial Guess	195.0	195.0	195.0	195.0	195.0
1	197.6	196.3	195.5	194.7	193.4
2	198.2	196.9	195.8	194.5	193.2
3	198.5	197.2	195.9	194.5	193.2
4	198.6	197.3	195.9	194.5	193.2
5	198.7	197.3	195.9	194.5	193.2
6	198.7	197.3	195.9	194.5	193.2
7	198.7	197.3	195.9	194.5	193.2

TWO-DIMENSIONAL STEADY HEAT CONDUCTION

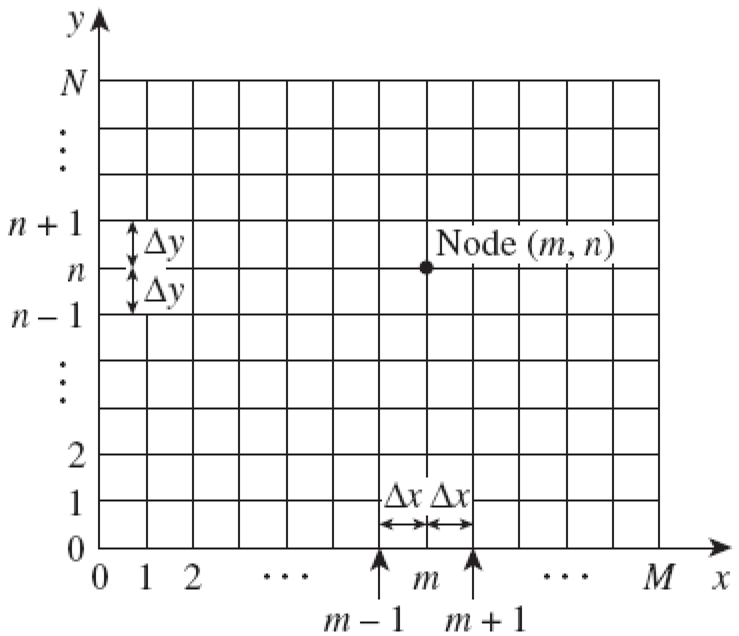


FIGURE 5–23

The nodal network for the finite difference formulation of two-dimensional conduction in rectangular coordinates.

Sometimes we need to consider heat transfer in other directions as well when the variation of temperature in other directions is significant.

We consider the numerical formulation and solution of two-dimensional steady heat conduction in rectangular coordinates using the finite difference method.

$$\begin{pmatrix}\text{Rate of heat conduction}\\ \text{at the left, top, right,}\\ \text{and bottom surfaces}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation inside}\\ \text{the element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change of}\\ \text{the energy content}\\ \text{of the element}\end{pmatrix}$$

$$\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, top}} + \dot{Q}_{\text{cond, right}} + \dot{Q}_{\text{cond, bottom}} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

$$A_x = \Delta y \times 1 = \Delta y \qquad A_y = \Delta x \times 1 = \Delta x$$

$$k\Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k\Delta y \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \dot{e}_{m,n}\,\Delta x\,\Delta y = 0$$

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\dot{e}_{m,n}}{k} = 0$$

For square mesh:

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{\dot{e}_{m,n} l^2}{k} = 0$$

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{e}_{\text{node}} l^2}{k} = 0$$

$$T_{\text{node}} = (T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}})/4$$

no heat generation

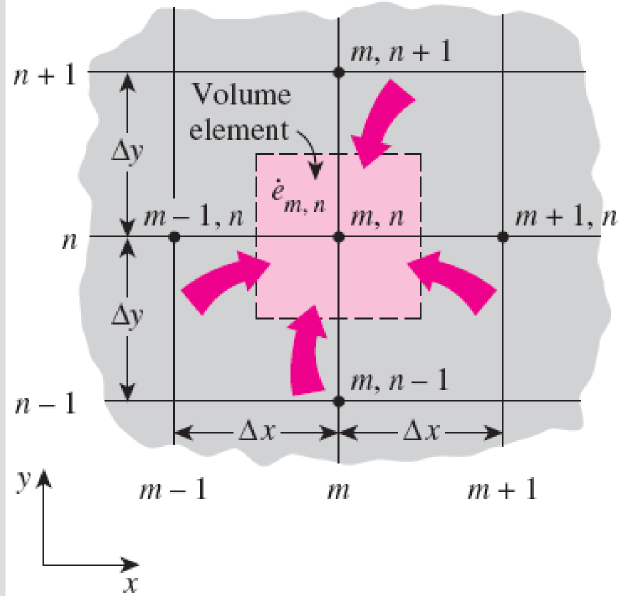


FIGURE 5–24

The volume element of a general interior node (m, n) for two-dimensional conduction in rectangular coordinates.

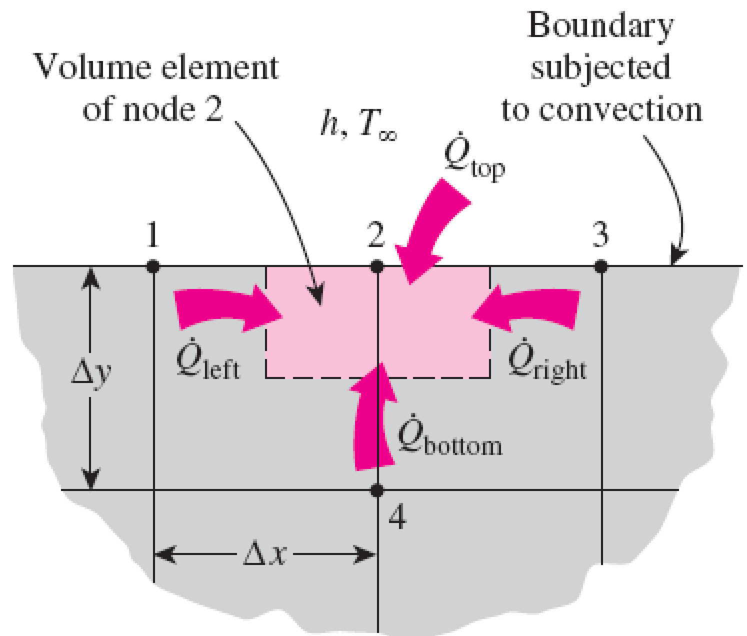
Boundary Nodes

The region is partitioned between the nodes by forming *volume elements* around the nodes, and an *energy balance* is written for each boundary node.

An *energy balance* on a volume element is

$$\sum_{\text{All sides}} \dot{Q} + \dot{e}V_{\text{element}} = 0$$

We assume, for convenience in formulation, all heat transfer to be *into* the volume element from all surfaces except for specified heat flux, whose direction is already specified.



$$\dot{Q}_{\text{left}} + \dot{Q}_{\text{top}} + \dot{Q}_{\text{right}} + \dot{Q}_{\text{bottom}} + \frac{\dot{e}_2 V_2}{k} = 0$$

FIGURE 5–25

The finite difference formulation of a boundary node is obtained by writing an energy balance on its volume element.

EXAMPLE 5–3 Steady Two-Dimensional Heat Conduction in L-Bars

Consider steady heat transfer in an L-shaped solid body whose cross section is given in Figure 5–26. Heat transfer in the direction normal to the plane of the paper is negligible, and thus heat transfer in the body is two-dimensional. The thermal conductivity of the body is $k = 15$ W/m · °C, and heat is generated in the body at a rate of $\dot{g} = 2 \times 10^6$ W/m^3. The left surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of 90°C. The entire top surface is subjected to convection to ambient air at $T_\infty = 25$°C with a convection coefficient of $h = 80$ W/m^2 · °C, and the right surface is subjected to heat flux at a uniform rate of $\dot{q}_R = 5000$ W/m^2. The nodal network of the problem consists of 15 equally spaced nodes with $\Delta x = \Delta y = 1.2$ cm, as shown in the figure. Five of the nodes are at the bottom surface, and thus their temperatures are known. Obtain the finite difference equations at the remaining nine nodes and determine the nodal temperatures by solving them.

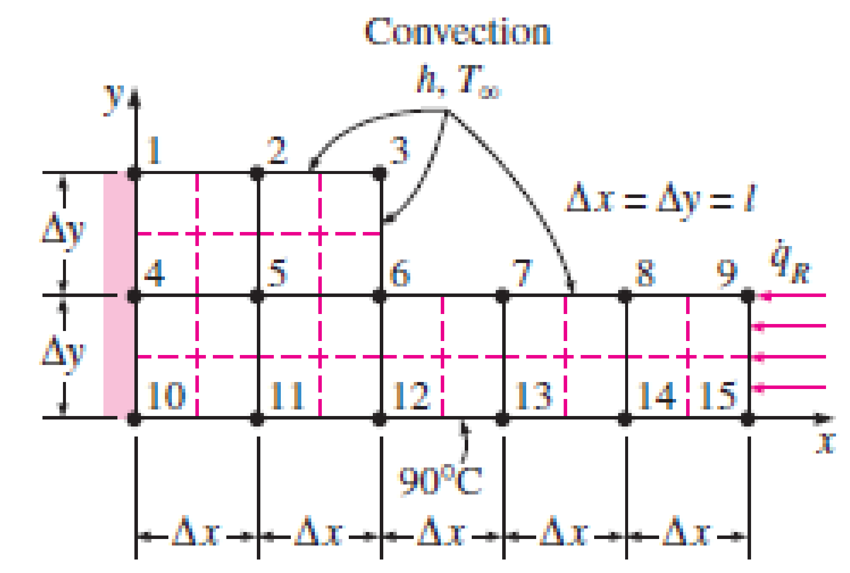
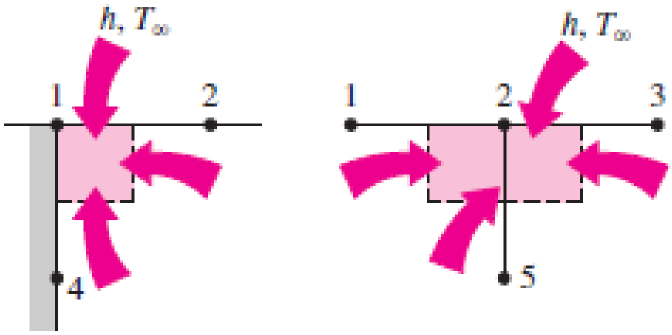


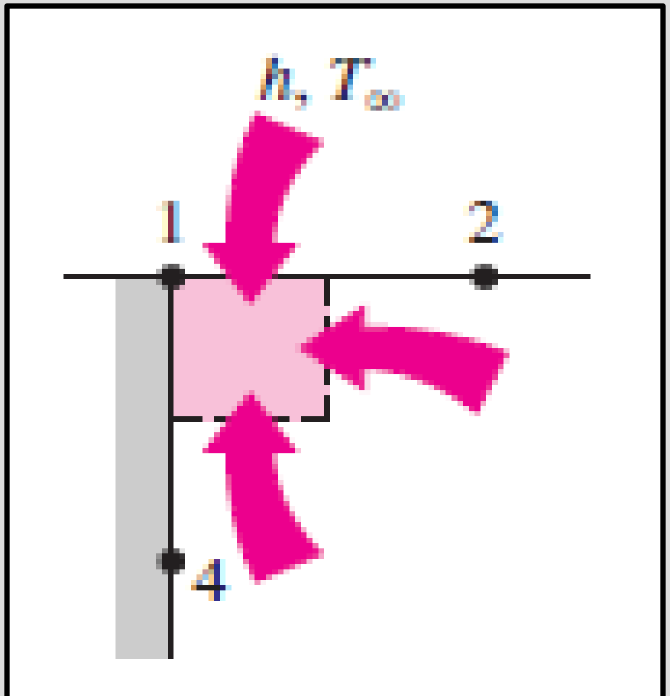
FIGURE 5–26
Schematic for Example 5–3 and the nodal network (the boundaries of volume elements of the nodes are indicated by dashed lines).



(a) Node 1 (b) Node 2

FIGURE 5–27

Schematics for energy balances on the volume elements of nodes 1 and 2.



(a) Node 1

SOLUTION Heat transfer in a long L-shaped solid bar with specified boundary conditions is considered. The nine unknown nodal temperatures are to be determined with the finite difference method.

Assumptions **1** Heat transfer is steady and two-dimensional, as stated. **2** Thermal conductivity is constant. **3** Heat generation is uniform. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity is given to be $k = 15$ W/m · °C.

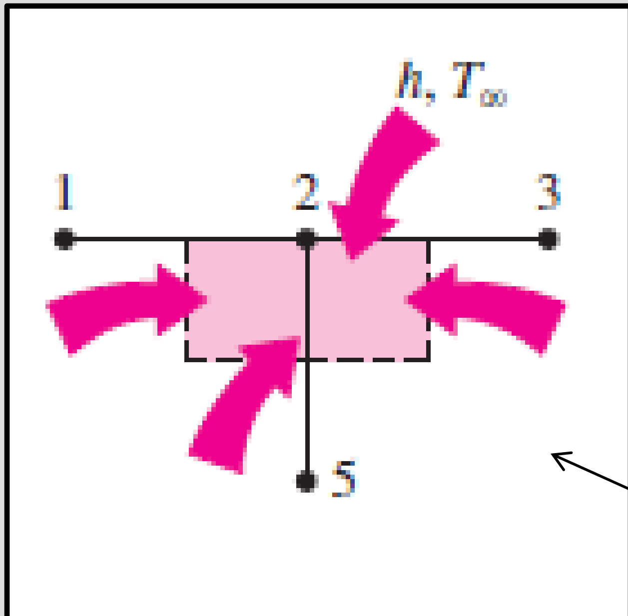
Analysis We observe that all nodes are boundary nodes except node 5, which is an interior node. Therefore, we will have to rely on energy balances to obtain the finite difference equations. But first we form the volume elements by partitioning the region among the nodes equitably by drawing dashed lines between the nodes. If we consider the volume element represented by an interior node to be *full size* (i.e., $\Delta x \times \Delta y \times 1$), then the element represented by a regular boundary node such as node 2 becomes *half size* (i.e., $\Delta x \times \Delta y/2 \times 1$), and a corner node such as node 1 is *quarter size* (i.e., $\Delta x/2 \times \Delta y/2 \times 1$). Keeping Eq. 5–36 in mind for the energy balance, the finite difference equations for each of the nine nodes are obtained as follows:

(a) *Node 1.* The volume element of this corner node is insulated on the left and subjected to convection at the top and to conduction at the right and bottom surfaces. An energy balance on this element gives [Fig. 5–27*a*]

$$0 + h\frac{\Delta x}{2}(T_\infty - T_1) + k\frac{\Delta y}{2}\frac{T_2 - T_1}{\Delta x} + k\frac{\Delta x}{2}\frac{T_4 - T_1}{\Delta y} + \dot{g}_1\frac{\Delta x}{2}\frac{\Delta y}{2} = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$-\left(2 + \frac{hl}{k}\right)T_1 + T_2 + T_4 = -\frac{hl}{k}T_\infty - \frac{\dot{g}_1 l^2}{2k}$$



(b) Node 2

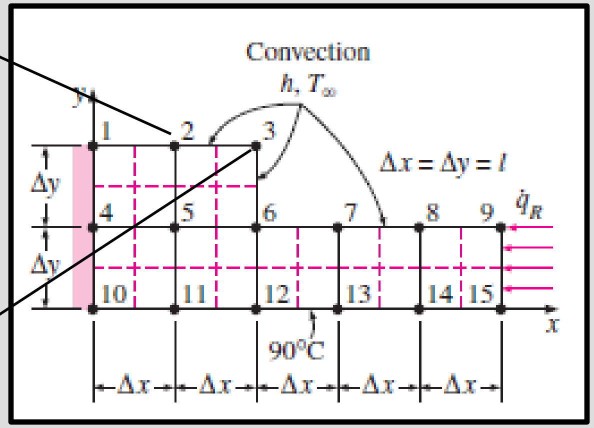
FIGURE 5–27

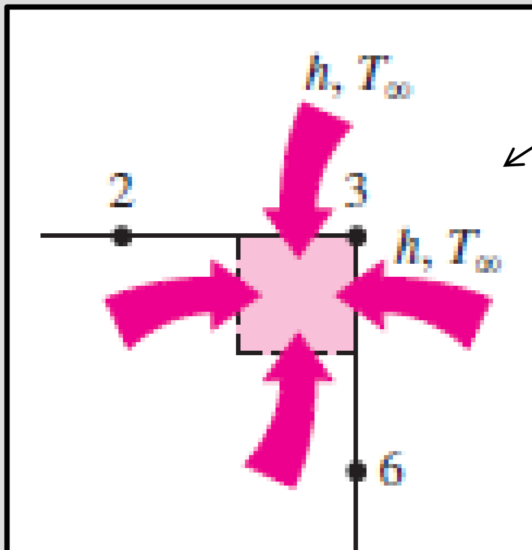
(*b*) *Node 2.* The volume element of this boundary node is subjected to convection at the top and to conduction at the right, bottom, and left surfaces. An energy balance on this element gives [Fig. 5–27*b*]

$$h\Delta x(T_\infty - T_2) + k\frac{\Delta y}{2}\frac{T_3 - T_2}{\Delta x} + k\Delta x\frac{T_5 - T_2}{\Delta y} + k\frac{\Delta y}{2}\frac{T_1 - T_2}{\Delta x} + \dot{g}_2\Delta x\frac{\Delta y}{2} = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$T_1 - \left(4 + \frac{2hl}{k}\right)T_2 + T_3 + 2T_5 = -\frac{2hl}{k}T_\infty - \frac{\dot{g}_2 l^2}{k}$$





(a) Node 3

FIGURE 5–28

(*c*) *Node 3.* The volume element of this corner node is subjected to convection at the top and right surfaces and to conduction at the bottom and left surfaces. An energy balance on this element gives [Fig. 5–28*a*]

$$h\left(\frac{\Delta x}{2} + \frac{\Delta y}{2}\right)(T_\infty - T_3) + k\frac{\Delta x}{2}\frac{T_6 - T_3}{\Delta y} + k\frac{\Delta y}{2}\frac{T_2 - T_3}{\Delta x} + \dot{g}_3\frac{\Delta x}{2}\frac{\Delta y}{2} = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

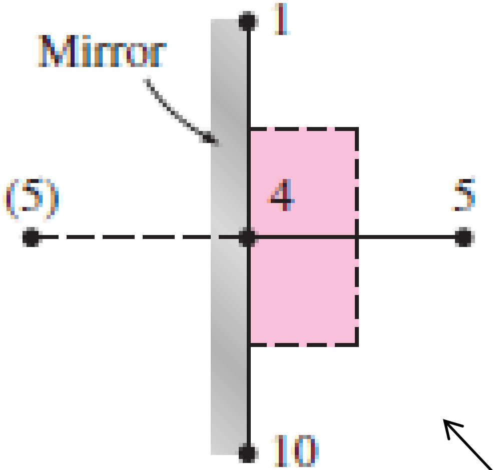
$$T_2 - \left(2 + \frac{2hl}{k}\right)T_3 + T_6 = -\frac{2hl}{k}T_\infty - \frac{\dot{g}_3 l^2}{2k}$$

(*d*) *Node 4*. This node is on the insulated boundary and can be treated as an interior node by replacing the insulation by a mirror. This puts a reflected image of node 5 to the left of node 4. Noting that $\Delta x = \Delta y = l$, the general interior node relation for the steady two-dimensional case (Eq. 5–35) gives [Fig. 5–28*b*]

$$T_5 + T_1 + T_5 + T_{10} - 4T_4 + \frac{\dot{g}_4 l^2}{k} = 0$$

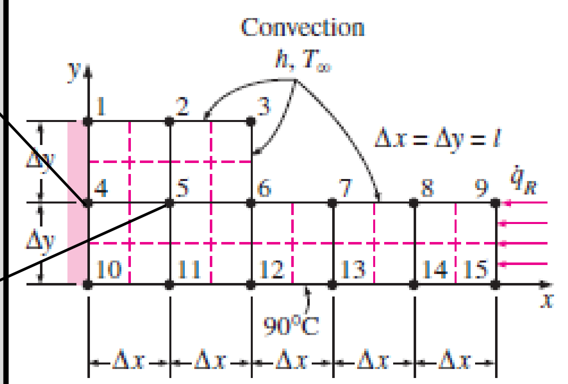
or, noting that $T_{10} = 90°$ C,

$$T_1 - 4T_4 + 2T_5 = -90 - \frac{\dot{g}_4 l^2}{k}$$



(*b*) Node 4

FIGURE 5–28

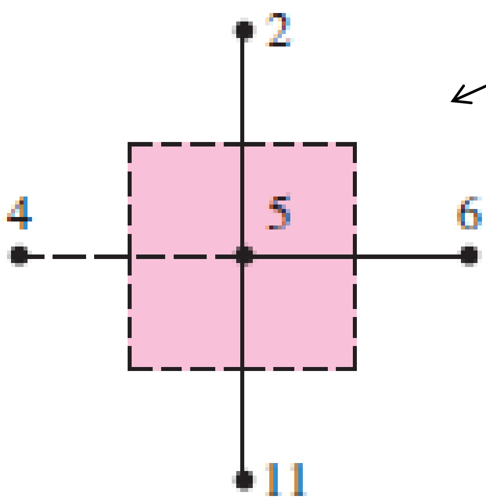


(*e*) Node 5. This is an interior node, and noting that $\Delta x = \Delta y = l$, the finite difference formulation of this node is obtained directly from Eq. 5–35 to be [Fig. 5–29*a*]

$$T_4 + T_2 + T_6 + T_{11} - 4T_5 + \frac{\dot{g}_5 l^2}{k} = 0$$

or, noting that $T_{11} = 90°$C,

$$T_2 + T_4 - 4T_5 + T_6 = -90 - \frac{\dot{g}_5 l^2}{k}$$



(*a*) Node 5

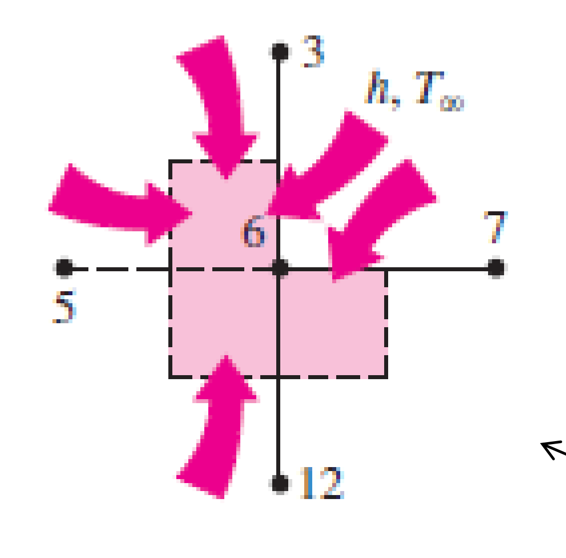
FIGURE 5–29

(f) *Node 6*. The volume element of this inner corner node is subjected to convection at the L-shaped exposed surface and to conduction at other surfaces. An energy balance on this element gives [Fig. 5–29*b*]

$$h\left(\frac{\Delta x}{2}+\frac{\Delta y}{2}\right)(T_\infty - T_6) + k\frac{\Delta y}{2}\frac{T_7-T_6}{\Delta x} + k\Delta x\frac{T_{12}-T_6}{\Delta y}$$
$$+ k\Delta y\frac{T_5-T_6}{\Delta x} + k\frac{\Delta x}{2}\frac{T_3-T_6}{\Delta y} + \dot g_6\frac{3\Delta x\Delta y}{4}=0$$

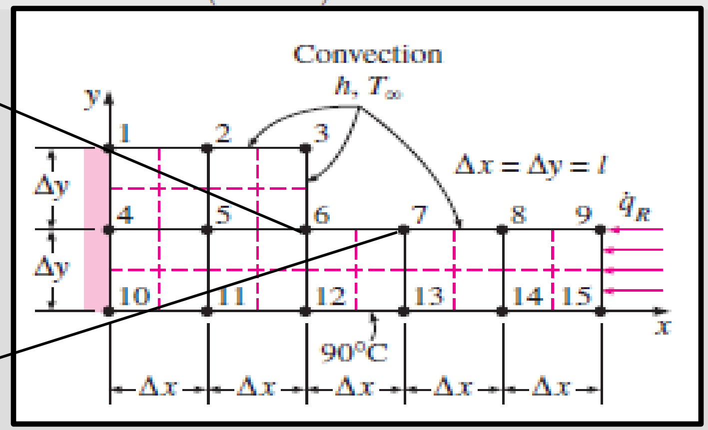
Taking $\Delta x = \Delta y = l$ and noting that $T_{12} = 90°C$, it simplifies to

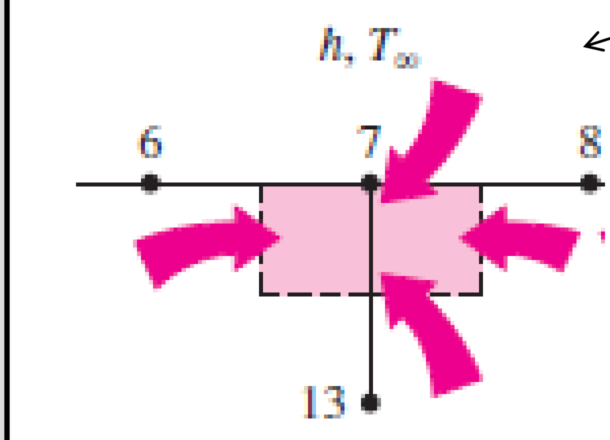
$$T_3+2T_5-\left(6+\frac{2hl}{k}\right)T_6+T_7 = -180-\frac{2hl}{k}T_\infty - \frac{3\dot g_6 l^2}{2k}$$



(*b*) Node 6

FIGURE 5–29





(*a*) Node 7.

FIGURE 5–30

(g) *Node 7*. The volume element of this boundary node is subjected to convection at the top and to conduction at the right, bottom, and left surfaces. An energy balance on this element gives [Fig. 5–30*a*]

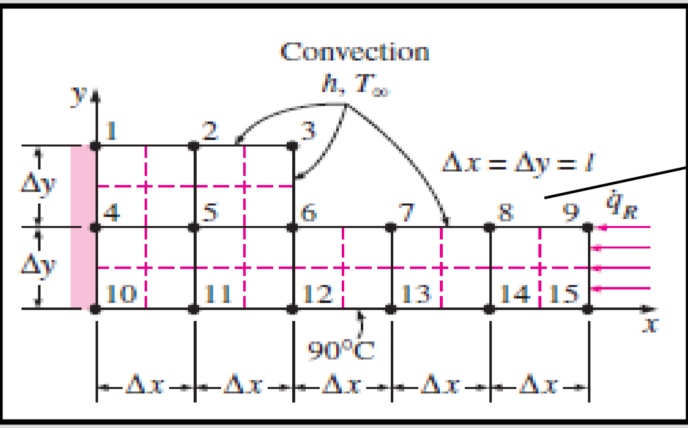
$$h\Delta x(T_\infty-T_7)+k\frac{\Delta y}{2}\frac{T_8-T_7}{\Delta x}+k\Delta x\frac{T_{13}-T_7}{\Delta y}$$
$$+k\frac{\Delta y}{2}\frac{T_6-T_7}{\Delta x}+\dot g_7\Delta x\frac{\Delta y}{2}=0$$

Taking $\Delta x = \Delta y = l$ and noting that $T_{13} = 90°C$, it simplifies to

$$T_6-\left(4+\frac{2hl}{k}\right)T_7+T_8=-180-\frac{2hl}{k}T_\infty-\frac{\dot g_7 l^2}{k}$$

(*h*) *Node 8*. This node is identical to Node 7, and the finite difference formulation of this node can be obtained from that of Node 7 by shifting the node numbers by 1 (i.e., replacing subscript m by $m + 1$). It gives

$$T_7 - \left(4 + \frac{2hl}{k}\right) T_8 + T_9 = -180 - \frac{2hl}{k} T_\infty - \frac{g_8 l^2}{k}$$



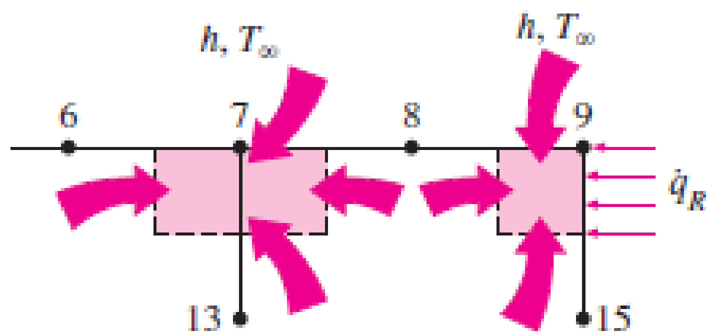


FIGURE 5–30

(*i*) *Node 9*. The volume element of this corner node is subjected to convection at the top surface, to heat flux at the right surface, and to conduction at the bottom and left surfaces. An energy balance on this element gives [Fig. 5–30*b*]

$$h \frac{\Delta x}{2}(T_\infty - T_9) + q_R \frac{\Delta y}{2} + k \frac{\Delta x}{2} \frac{T_{15} - T_9}{\Delta y} + k \frac{\Delta y}{2} \frac{T_8 - T_9}{\Delta x} + g_9 \frac{\Delta x}{2} \frac{\Delta y}{2} = 0$$

Taking $\Delta x = \Delta y = l$ and noting that $T_{15} = 90°C$, it simplifies to

$$T_8 - \left(2 + \frac{hl}{k}\right) T_9 = -90 - \frac{q_R l}{k} - \frac{hl}{k} T_\infty - \frac{g_9 l^2}{2k}$$

This completes the development of finite difference formulation for this problem. Substituting the given quantities, the system of nine equations for the determination of nine unknown nodal temperatures becomes

$$
\begin{aligned}
-2.064T_1 + T_2 + T_4 &= -11.2 \\
T_1 - 4.128T_2 + T_3 + 2T_5 &= -22.4 \\
T_2 - 2.128T_3 + T_6 &= -12.8 \\
T_1 - 4T_4 + 2T_5 &= -109.2 \\
T_2 + T_4 - 4T_5 + T_6 &= -109.2 \\
T_3 + 2T_5 - 6.128T_6 + T_7 &= -212.0 \\
T_6 - 4.128T_7 + T_8 &= -202.4 \\
T_7 - 4.128T_8 + T_9 &= -202.4 \\
T_8 - 2.064T_9 &= -105.2
\end{aligned}
$$

which is a system of nine algebraic equations with nine unknowns. Using an equation solver, its solution is determined to be

$$
\begin{array}{lll}
T_1 = 112.1^\circ\text{C} & T_2 = 110.8^\circ\text{C} & T_3 = 106.6^\circ\text{C} \\
T_4 = 109.4^\circ\text{C} & T_5 = 108.1^\circ\text{C} & T_6 = 103.2^\circ\text{C} \\
T_7 = \ \ 97.3^\circ\text{C} & T_8 = \ \ 96.3^\circ\text{C} & T_9 = \ \ 97.6^\circ\text{C}
\end{array}
$$

Note that the temperature is the highest at node 1 and the lowest at node 8. This is consistent with our expectations since node 1 is the farthest away from the bottom surface, which is maintained at 90°C and has one side insulated, and node 8 has the largest exposed area relative to its volume while being close to the surface at 90°C.

Irregular Boundaries

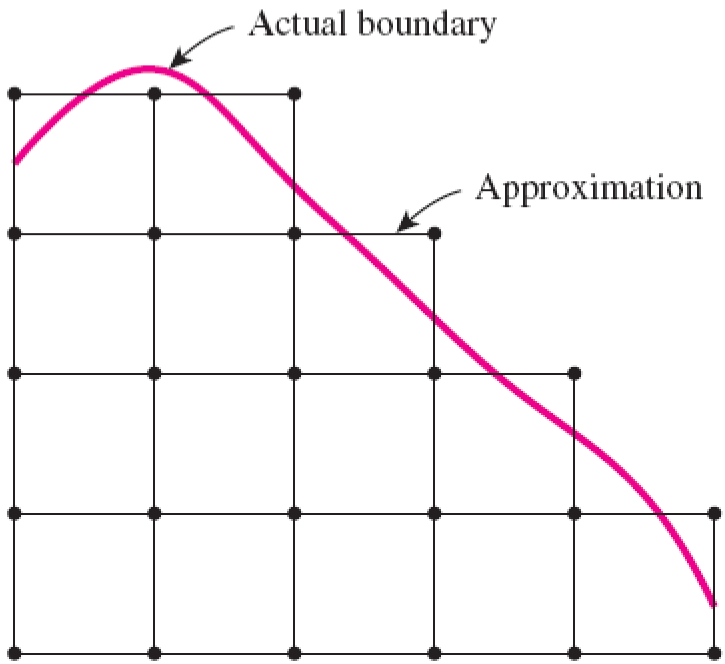


FIGURE 5–31

Approximating an irregular boundary with a rectangular mesh.

Many geometries encountered in practice such as turbine blades or engine blocks do not have simple shapes, and it is difficult to fill such geometries having irregular boundaries with simple volume elements.

A practical way of dealing with such geometries is to replace the irregular geometry by a series of simple volume elements.

This simple approach is often satisfactory for practical purposes, especially when the nodes are closely spaced near the boundary.

More sophisticated approaches are available for handling irregular boundaries, and they are commonly incorporated into the commercial software packages.

EXAMPLE 5–4 Heat Loss through Chimneys

Hot combustion gases of a furnace are flowing through a square chimney made of concrete (k = 1.4 W/m · °C). The flow section of the chimney is 20 cm × 20 cm, and the thickness of the wall is 20 cm. The average temperature of the hot gases in the chimney is T_i = 300°C, and the average convection heat transfer coefficient inside the chimney is h_i = 70 W/m^2 · °C. The chimney is losing heat from its outer surface to the ambient air at T_o = 20°C by convection with a heat transfer coefficient of h_o = 21 W/m^2 · °C and to the sky by radiation. The emissivity of the outer surface of the wall is ε = 0.9, and the effective sky temperature is estimated to be 260 K. Using the finite difference method with $\Delta x = \Delta y$ = 10 cm and taking full advantage of symmetry, determine the temperatures at the nodal points of a cross section and the rate of heat loss for a 1-m-long section of the chimney.

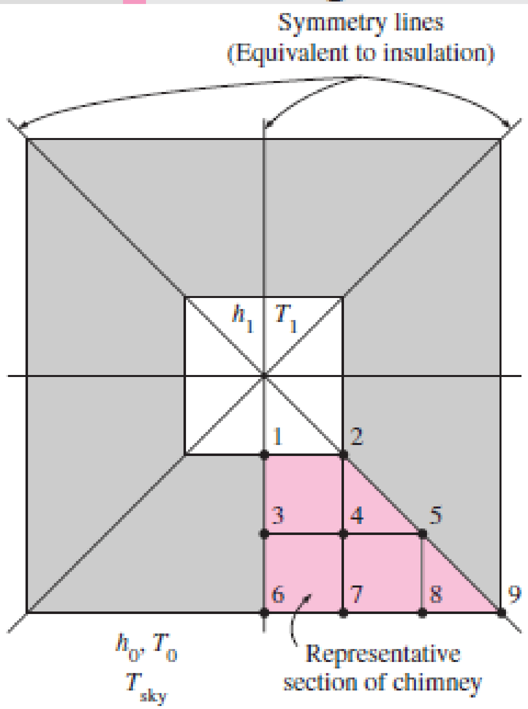


FIGURE 5–32

Schematic of the chimney discussed in Example 5–4 and the nodal network for a representative section.

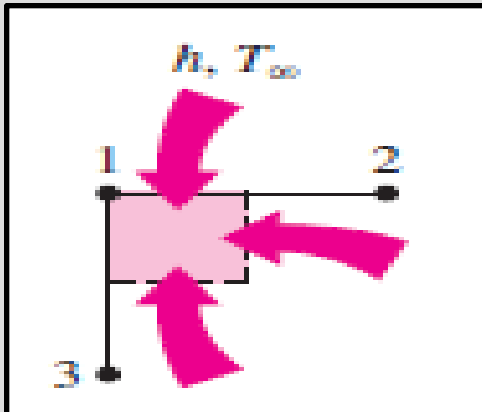
SOLUTION Heat transfer through a square chimney is considered. The nodal temperatures and the rate of heat loss per unit length are to be determined with the finite difference method.

Assumptions **1** Heat transfer is steady since there is no indication of change with time. **2** Heat transfer through the chimney is two-dimensional since the height of the chimney is large relative to its cross section, and thus heat conduction through the chimney in the axial direction is negligible. It is tempting to simplify the problem further by considering heat transfer in each wall to be one-dimensional, which would be the case if the walls were thin and thus the corner effects were negligible. This assumption cannot be justified in this case since the walls are very thick and the corner sections constitute a considerable portion of the chimney structure. **3** Thermal conductivity is constant.

Properties The properties of chimney are given to be $k = 1.4$ W/m · °C and $\varepsilon = 0.9$.

Analysis The cross section of the chimney is given in Figure 5–32. The most striking aspect of this problem is the apparent symmetry about the horizontal and vertical lines passing through the midpoint of the chimney as well as the diagonal axes, as indicated on the figure. Therefore, we need to consider only one-eighth of the geometry in the solution whose nodal network consists of nine equally spaced nodes.

No heat can cross a symmetry line, and thus symmetry lines can be treated as insulated surfaces and thus "mirrors" in the finite difference formulation. Then the nodes in the middle of the symmetry lines can be treated as interior nodes by using mirror images. Six of the nodes are boundary nodes, so we will have to write energy balances to obtain their finite difference formulations. First we partition the region among the nodes equitably by drawing dashed lines between the nodes through the middle. Then the region around a node surrounded by the boundary or the dashed lines represents the volume element of the node. Considering a unit depth and using the energy balance approach for the boundary nodes (again assuming all heat transfer into the volume element for convenience) and the formula for the interior nodes, the finite difference equations for the nine nodes are determined as follows:



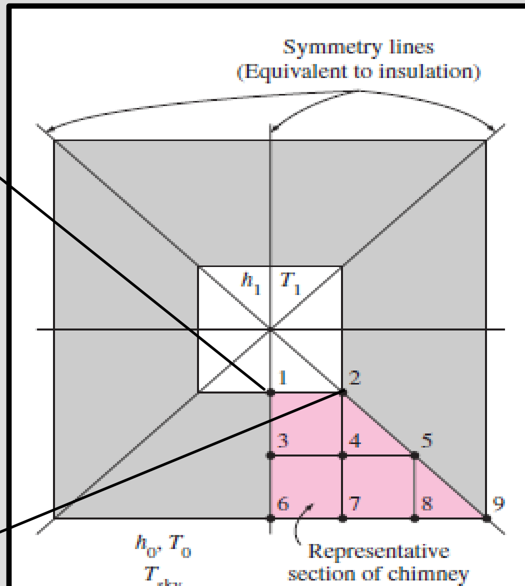
(a) Node 1

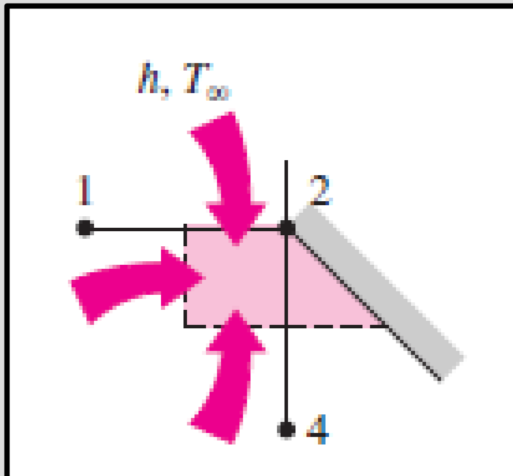
(a) Node 1. On the inner boundary, subjected to convection, Figure 5–33a

$$0 + h_i \frac{\Delta x}{2}(T_i - T_1) + k\frac{\Delta y}{2}\frac{T_2 - T_1}{\Delta x} + k\frac{\Delta x}{2}\frac{T_3 - T_1}{\Delta y} + 0 = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$-\left(2 + \frac{h_i l}{k}\right)T_1 + T_2 + T_3 = -\frac{h_i l}{k}T_i$$





(b) Node 2

(b) Node 2. On the inner boundary, subjected to convection, Figure 5–33b

$$k\frac{\Delta y}{2}\frac{T_1 - T_2}{\Delta x} + h_i\frac{\Delta x}{2}(T_i - T_2) + 0 + k\Delta x\frac{T_4 - T_2}{\Delta y} = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$T_1 - \left(3 + \frac{h_i l}{k}\right)T_2 + 2T_4 = -\frac{h_i l}{k}T_i$$

(*c*) Nodes 3, 4, *and* 5. (Interior nodes, Fig. 5–34)

$$\text{Node 3: } T_4 + T_1 + T_4 + T_6 - 4T_3 = 0$$
$$\text{Node 4: } T_3 + T_2 + T_5 + T_7 - 4T_4 = 0$$
$$\text{Node 5: } T_4 + T_4 + T_8 + T_8 - 4T_5 = 0$$

(*d*) Node 6. (On the outer boundary, subjected to convection and radiation)

$$0 + k\frac{\Delta x}{2}\frac{T_3 - T_6}{\Delta y} + k\frac{\Delta y}{2}\frac{T_7 - T_6}{\Delta x}$$
$$+ h_o\frac{\Delta x}{2}(T_o - T_6) + \varepsilon\sigma\frac{\Delta x}{2}(T_{\text{sky}}^4 - T_6^4) = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$T_2 + T_3 - \left(2 + \frac{h_o l}{k}\right)T_6 = -\frac{h_o l}{k}T_o - \frac{\varepsilon\sigma l}{k}(T_{\text{sky}}^4 - T_6^4)$$

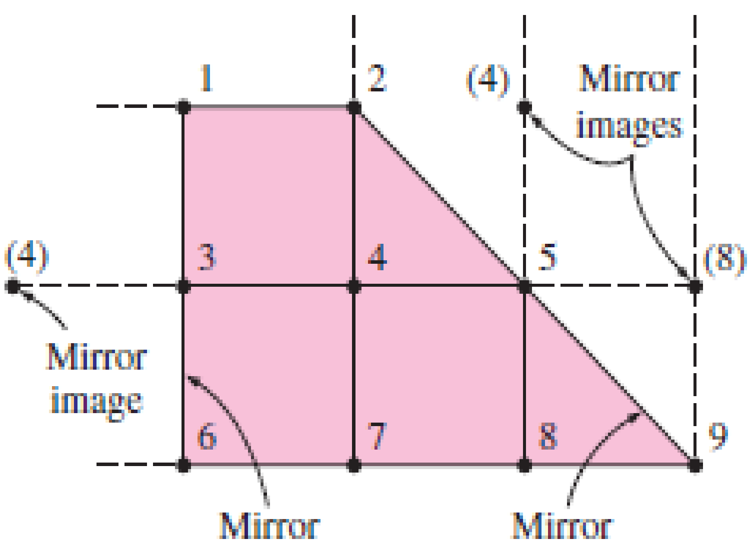
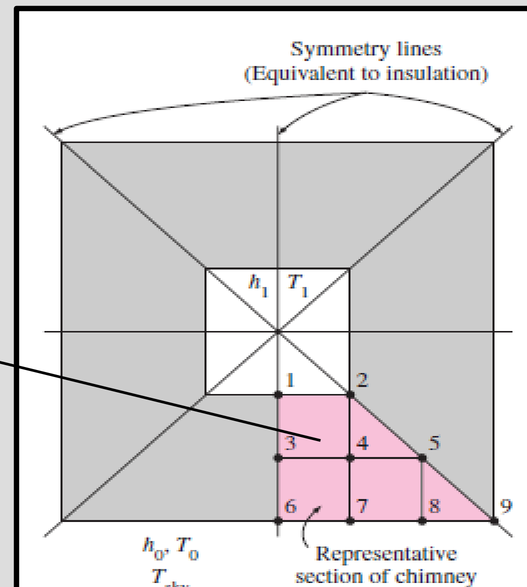
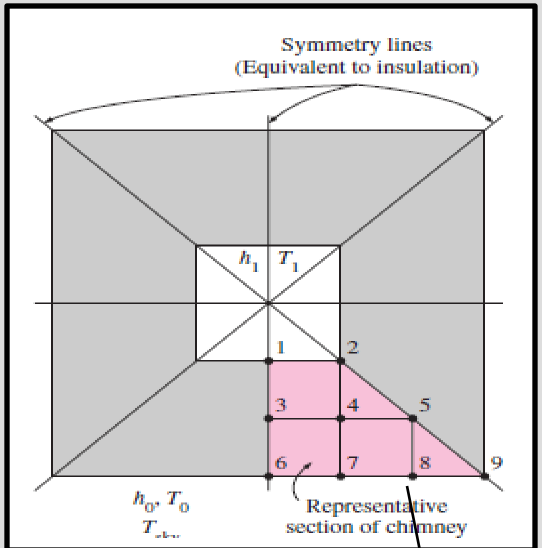


FIGURE 5–34





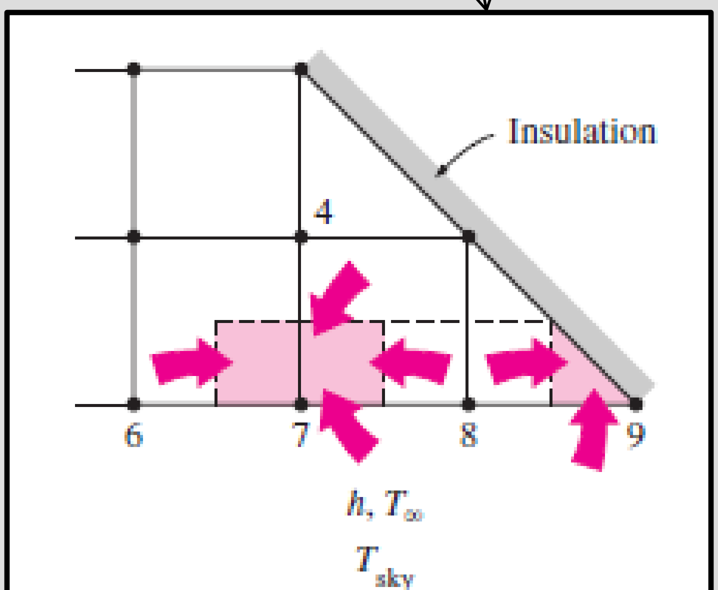


FIGURE 5–35

(*e*) Node 7. (On the outer boundary, subjected to convection and radiation, Fig. 5–35)

$$k\frac{\Delta y}{2}\frac{T_6 - T_7}{\Delta x} + k\Delta x\frac{T_4 - T_7}{\Delta y} + k\frac{\Delta y}{2}\frac{T_8 - T_7}{\Delta x} + h_o\Delta x(T_o - T_7) + \varepsilon\sigma\Delta x(T_{sky}^4 - T_7^4) = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$2T_4 + T_6 - \left(4 + \frac{2h_o l}{k}\right)T_7 + T_8 = -\frac{2h_o l}{k}T_o - \frac{2\varepsilon\sigma l}{k}(T_{sky}^4 - T_7^4)$$

(*f*) Node 8. Same as Node 7, except shift the node numbers up by 1 (replace 4 by 5, 6 by 7, 7 by 8, and 8 by 9 in the last relation)

$$2T_5 + T_7 - \left(4 + \frac{2h_o l}{k}\right)T_8 + T_9 = -\frac{2h_o l}{k}T_o - \frac{2\varepsilon\sigma l}{k}(T_{sky}^4 - T_8^4)$$

(*g*) Node 9. (On the outer boundary, subjected to convection and radiation, Fig. 5–35)

$$k\frac{\Delta y}{2}\frac{T_8 - T_9}{\Delta x} + 0 + h_o\frac{\Delta x}{2}(T_o - T_9) + \varepsilon\sigma\frac{\Delta x}{2}(T_{sky}^4 - T_9^4) = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$T_8 - \left(1 + \frac{h_o l}{k}\right) T_9 = -\frac{h_o l}{k} T_o - \frac{\varepsilon \sigma l}{k} (T_{\text{sky}}^4 - T_9^4)$$

This problem involves radiation, which requires the use of absolute temperature, and thus all temperatures should be expressed in Kelvin. Alternately, we could use °C for all temperatures provided that the four temperatures in the radiation terms are expressed in the form $(T + 273)^4$. Substituting the given quantities, the system of nine equations for the determination of nine unknown nodal temperatures in a form suitable for use with the Gauss-Seidel iteration method becomes

$$\begin{aligned}
T_1 &= (T_2 + T_3 + 2865)/7 \\
T_2 &= (T_1 + 2T_4 + 2865)/8 \\
T_3 &= (T_1 + 2T_4 + T_6)/4 \\
T_4 &= (T_2 + T_3 + T_5 + T_7)/4 \\
T_5 &= (2T_4 + 2T_8)/4 \\
T_6 &= (T_2 + T_3 + 456.2 - 0.3645 \times 10^{-9}\, T_6^4)/3.5 \\
T_7 &= (2T_4 + T_6 + T_8 + 912.4 - 0.729 \times 10^{-9}\, T_7^4)/7 \\
T_8 &= (2T_5 + T_7 + T_9 + 912.4 - 0.729 \times 10^{-9}\, T_8^4)/7 \\
T_9 &= (T_8 + 456.2 - 0.3645 \times 10^{-9}\, T_9^4)/2.5
\end{aligned}$$

which is a system of *nonlinear* equations. Using an equation solver, its solution is determined to be

$$\begin{aligned}
&T_1 = 545.7 \text{ K} = 272.6°\text{C} \quad && T_2 = 529.2 \text{ K} = 256.1°\text{C} \quad && T_3 = 425.2 \text{ K} = 152.1°\text{C} \\
&T_4 = 411.2 \text{ K} = 138.0°\text{C} \quad && T_5 = 362.1 \text{ K} = 89.0°\text{C} \quad && T_6 = 332.9 \text{ K} = 59.7°\text{C} \\
&T_7 = 328.1 \text{ K} = 54.9°\text{C} \quad && T_8 = 313.1 \text{ K} = 39.9°\text{C} \quad && T_9 = 296.5 \text{ K} = 23.4°\text{C}
\end{aligned}$$

The variation of temperature in the chimney is shown in Figure 5–36.

Note that the temperatures are highest at the inner wall (but less than 300°C) and lowest at the outer wall (but more that 260 K), as expected.

The average temperature at the outer surface of the chimney weighed by the surface area is

$$T_{\text{wall, out}} = \frac{(0.5T_6 + T_7 + T_8 + 0.5T_9)}{(0.5 + 1 + 1 + 0.5)}$$

$$= \frac{0.5 \times 332.9 + 328.1 + 313.1 + 0.5 \times 296.5}{3} = 318.6\text{ K}$$

Then the rate of heat loss through the 1-m-long section of the chimney can be determined approximately from

$$\begin{aligned}\dot{Q}_{\text{chimney}} &= h_o A_o (T_{\text{wall, out}} - T_o) + \varepsilon\sigma A_o (T^4_{\text{wall, out}} - T^4_{\text{sky}})\\ &= (21\text{ W/m}^2\cdot\text{K})[4 \times (0.6\text{ m})(1\text{ m})](318.6 - 293)\text{K}\\ &\quad + 0.9(5.67 \times 10^{-8}\text{ W/m}^2\cdot\text{K}^4)\\ &\qquad [4 \times (0.6\text{ m})(1\text{ m})](318.6\text{ K})^4 - (260\text{ K})^4]\\ &= 1291 + 702 = \mathbf{1993\text{ W}}\end{aligned}$$

We could also determine the heat transfer by finding the average temperature of the inner wall, which is (272.6 + 256.1)/2 = 264.4°C, and applying Newton's law of cooling at that surface:

$$\begin{aligned}\dot{Q}_{\text{chimney}} &= h_i A_i (T_i - T_{\text{wall, in}})\\ &= (70\text{ W/m}^2\cdot\text{K})[4 \times (0.2\text{ m})(1\text{ m})](300 - 264.4)°\text{C} = 1994\text{ W}\end{aligned}$$

The difference between the two results is due to the approximate nature of the numerical analysis.

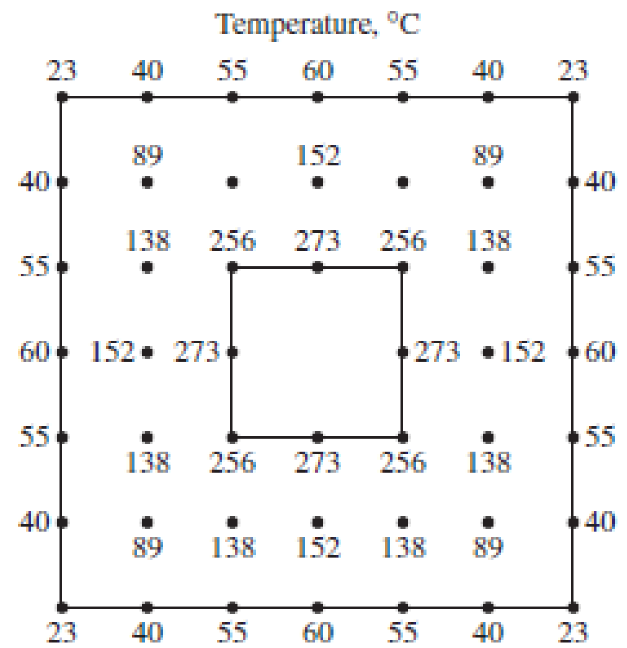


FIGURE 5–36
The variation of temperature in the chimney.

TRANSIENT HEAT CONDUCTION

The finite difference solution of transient problems requires *discretization in time* in addition to discretization in space.

This is done by selecting a suitable time step Δt and solving for the unknown nodal temperatures repeatedly for each Δt until the solution at the desired time is obtained.

In transient problems, the *superscript i* is used as the *index* or *counter* of time steps, with $i = 0$ corresponding to the specified initial condition.

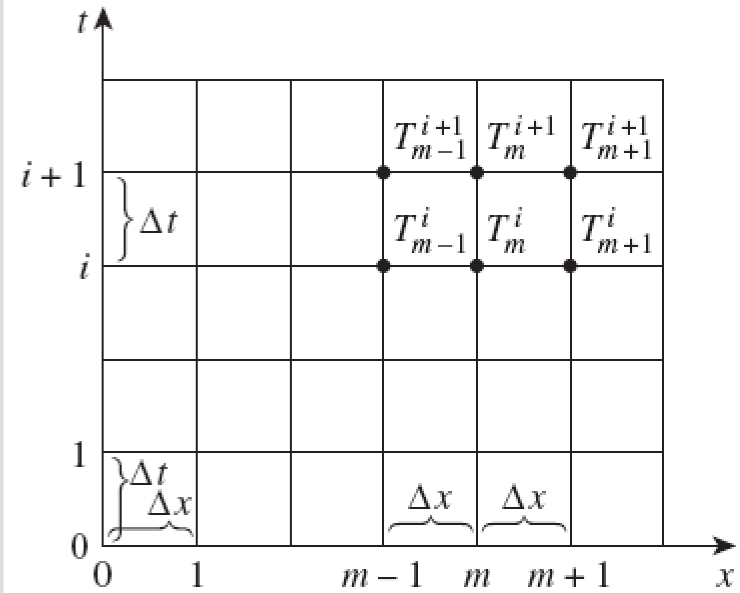


FIGURE 5–37

Finite difference formulation of time-dependent problems involves discrete points in time as well as space.

$$\begin{pmatrix}\text{Heat transferred into}\\ \text{the volume element}\\ \text{from all of its surfaces}\\ \text{during } \Delta t\end{pmatrix} + \begin{pmatrix}\text{Heat generated}\\ \text{within the}\\ \text{volume element}\\ \text{during } \Delta t\end{pmatrix} = \begin{pmatrix}\text{The change in the}\\ \text{energy content of}\\ \text{the volume element}\\ \text{during } \Delta t\end{pmatrix}$$

$$\Delta t \times \sum_{\text{All sides}} \dot{Q} + \Delta t \times \dot{E}_{\text{gen, element}} = \Delta E_{\text{element}}$$

$$\Delta E_{\text{element}} = mc_p \Delta T = \rho V_{\text{element}}\, c_p \Delta T,$$

$$\sum_{\text{All sides}} \dot{Q} + \dot{E}_{\text{gen, element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = \rho V_{\text{element}}\, c_p \frac{\Delta T}{\Delta t}$$

$$\sum_{\text{All sides}} \dot{Q} + \dot{E}_{\text{gen, element}} = \rho V_{\text{element}}\, c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Explicit method: If temperatures at the *previous* time step ***i*** is used.

Implicit method: If temperatures at the *new* time step ***i* + 1** is used.

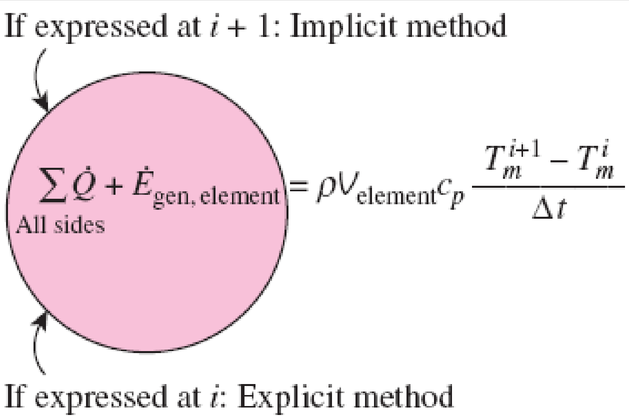
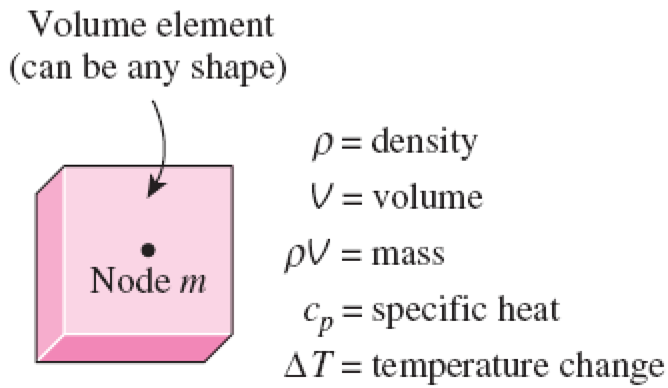


FIGURE 5–39

The formulation of explicit and implicit methods differs at the time step (previous or new) at which the heat transfer and heat generation terms are expressed.



$$\Delta U = \rho V c_p \Delta T = \rho V c_p (T_m^{i+1} - T_m^i)$$

FIGURE 5–38

The change in the energy content of the volume element of a node during a time interval Δt.

$$\sum_{\text{All sides}} \dot{Q} + \dot{E}_{\text{gen, element}} = \rho V_{\text{element}} \, c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Explicit method:
$$\sum_{\text{All sides}} \dot{Q}^i + \dot{E}^i_{\text{gen, element}} = \rho V_{\text{element}} \, c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Implicit method:
$$\sum_{\text{All sides}} \dot{Q}^{i+1} + \dot{E}^{i+1}_{\text{gen, element}} = \rho V_{\text{element}} \, c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Transient Heat Conduction in a Plane Wall

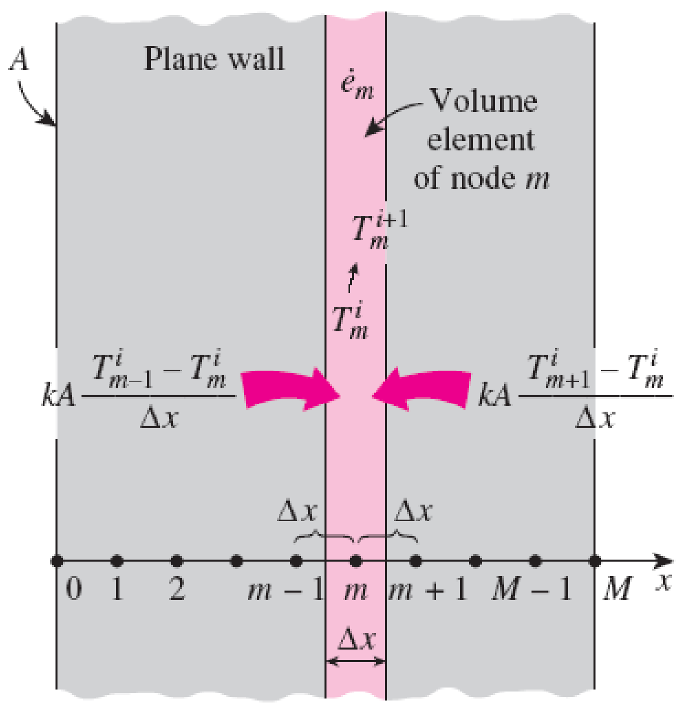


FIGURE 5–40

The nodal points and volume elements for the transient finite difference formulation of one-dimensional conduction in a plane wall.

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + \dot{e}_m A\Delta x = \rho A\Delta x c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

$$T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{e}_m \Delta x^2}{k} = \frac{\Delta x^2}{\alpha\Delta t}(T_m^{i+1} - T_m^i)$$

$$\alpha = k/\rho c_p$$

$$\tau = \frac{\alpha\Delta t}{\Delta x^2}$$

mesh Fourier number

$$T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{e}_m \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{e}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau} \qquad \text{(explicit)}$$

$$T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1 - 2\tau)\,T_m^i + \tau\frac{\dot{e}_m^i \Delta x^2}{k}$$

$$T_{m-1}^{i+1} - 2T_m^{i+1} + T_{m+1}^{i+1} + \frac{\dot{e}_m^{i+1} \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau} \qquad \text{(implicit)}$$

$$\tau T_{m-1}^{i+1} - (1 + 2\tau)\,T_m^{i+1} + \tau T_{m+1}^{i+1} + \tau\frac{\dot{e}_m^{i+1} \Delta x^2}{k} + T_m^i = 0$$

$$hA(T_\infty - T_0^i) + kA\frac{T_1^i - T_0^i}{\Delta x} + \dot{e}_0^i\, A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2}c_p\frac{T_0^{i+1} - T_0^i}{\Delta t}$$

$$T_0^{i+1} = \left(1 - 2\tau - 2\tau\frac{h\Delta x}{k}\right)T_0^i + 2\tau T_1^i + 2\tau\frac{h\Delta x}{k}T_\infty + \tau\frac{\dot{e}_0^i\,\Delta x^2}{k}$$

No heat generation and $\tau = 0.5$

$$(T_m^{i+1} = (T_{m-1}^i + T_{m+1}^i)/2,$$

The temperature of an interior node *at the new time step is simply the average of the temperatures of its neighboring* nodes *at the previous time step.*

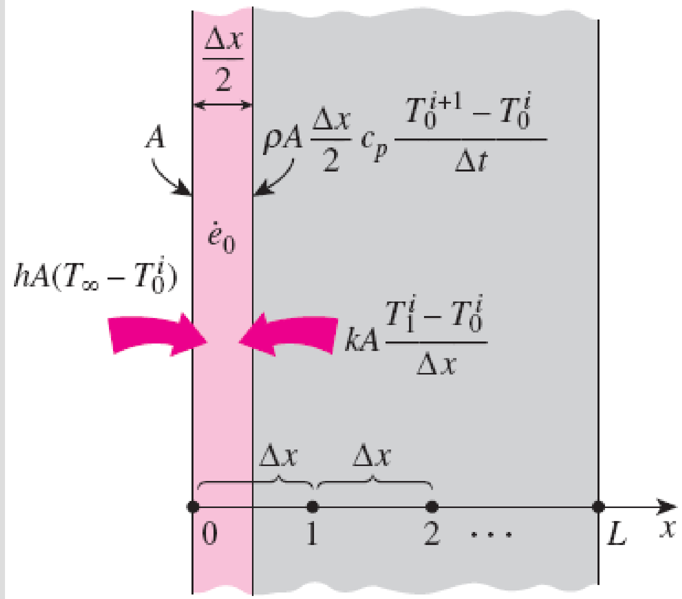


FIGURE 5–41

Schematic for the explicit finite difference formulation of the convection condition at the left boundary of a plane wall.

Stability Criterion for Explicit Method: Limitation on Δt

The explicit method is easy to use, but it suffers from an undesirable feature that severely restricts its utility: the explicit method is not unconditionally stable, and the largest permissible value of the time step Δt is limited by the stability criterion.

If the time step Δt is not sufficiently small, the solutions obtained by the explicit method may oscillate wildly and diverge from the actual solution.

To avoid such divergent oscillations in nodal temperatures, the value of Δt must be maintained below a certain upper limit established by the **stability criterion**.

Explicit formulation:

$$T_0^{i+1} = a_0 T_0^i + \cdots$$
$$T_1^{i+1} = a_1 T_1^i + \cdots$$
$$\vdots$$
$$T_m^{i+1} = a_m T_m^i + \cdots$$
$$\vdots$$
$$T_M^{i+1} = a_M T_M^i + \cdots$$

Stability criterion:

$$a_m \geq 0, \; m = 0, 1, 2, \ldots m, \ldots M$$

FIGURE 5–42

The stability criterion of the explicit method requires all primary coefficients to be positive or zero.

$$\tau = \frac{\alpha \Delta t}{\Delta x^2} \leq \frac{1}{2} \qquad \left(\begin{array}{c}\text{interior nodes, one-dimensional heat} \\ \text{transfer in rectangular coordinates}\end{array}\right)$$

$$\Delta t \leq \frac{1}{2}\frac{\Delta x^2}{\alpha} = \frac{(0.01 \text{ m})^2}{2(0.45 \times 10^{-6} \text{ m}^2/\text{s})} = 111 \text{ s} = 1.85 \text{ min}$$

Example

$$1 - 2\tau - 2\tau \frac{h\Delta x}{k} \geqslant 0 \qquad \text{or} \qquad \tau \leq \frac{1}{2(1 + h\Delta x/k)}$$

$$T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1 - 2\tau)T_m^i$$

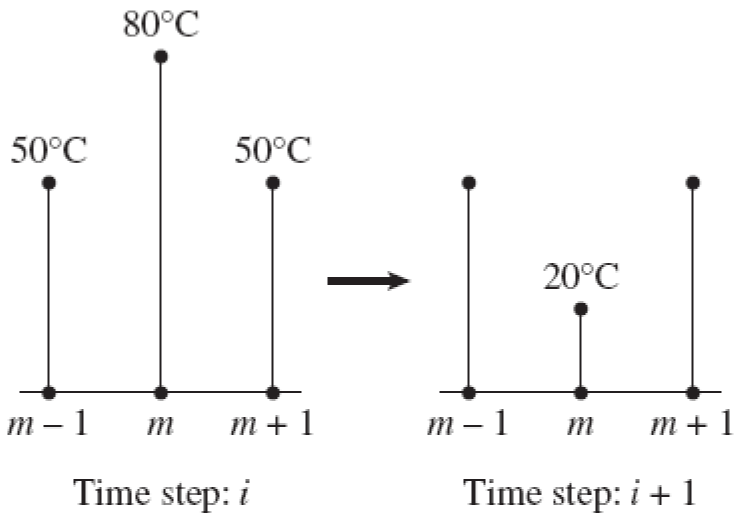


FIGURE 5–43

The violation of the stability criterion in the explicit method may result in the violation of the second law of thermodynamics and thus divergence of solution.

The implicit method is *unconditionally stable*, and thus we can use any time step we please with that method (of course, the smaller the time step, the better the accuracy of the solution).

The disadvantage of the implicit method is that it results in a set of equations that must be solved *simultaneously* for each time step.

Both methods are used in practice.

EXAMPLE 5–5 **Transient Heat Conduction in a Large Uranium Plate**

Consider a large uranium plate of thickness $L = 4$ cm, thermal conductivity $k = 28$ W/m · °C, and thermal diffusivity $\alpha = 12.5 \times 10^{-6}$ m²/s that is initially at a uniform temperature of 200°C. Heat is generated uniformly in the plate at a constant rate of $\dot{g} = 5 \times 10^6$ W/m³. At time $t = 0$, one side of the plate is brought into contact with iced water and is maintained at 0°C at all times, while the other side is subjected to convection to an environment at $T_\infty = 30$°C with a heat transfer coefficient of $h = 45$ W/m² · °C, as shown in Figure 5–44. Considering a total of three equally spaced nodes in the medium, two at the boundaries and one at the middle, estimate the exposed surface temperature of the plate 2.5 min after the start of cooling using (*a*) the explicit method and (*b*) the implicit method.

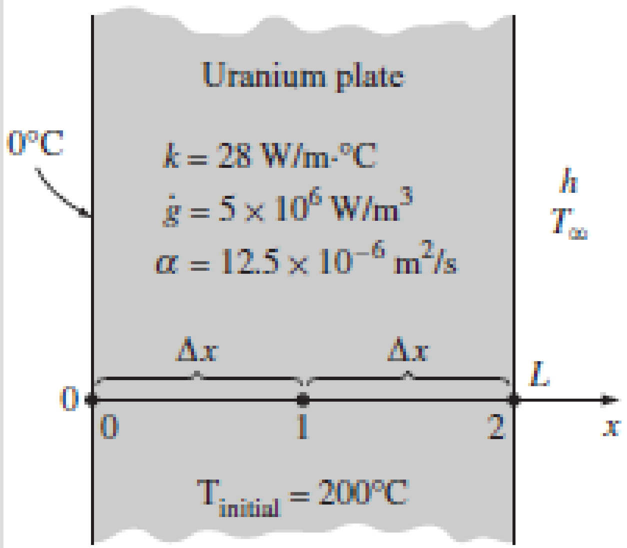


FIGURE 5–44
Schematic for Example 5–5.

SOLUTION We have solved this problem in Example 5–1 for the steady case, and here we repeat it for the transient case to demonstrate the application of the transient finite difference methods. Again we assume one-dimensional heat transfer in rectangular coordinates and constant thermal conductivity. The number of nodes is specified to be $M = 3$, and they are chosen to be at the two surfaces of the plate and at the middle, as shown in the figure. Then the nodal spacing Δx becomes

$$\Delta x = \frac{L}{M-1} = \frac{0.04 \text{ m}}{3-1} = 0.02 \text{ m}$$

We number the nodes as 0, 1, and 2. The temperature at node 0 is given to be $T_0 = 0°C$ at all times, and the temperatures at nodes 1 and 2 are to be determined. This problem involves only two unknown nodal temperatures, and thus we need to have only two equations to determine them uniquely. These equations are obtained by applying the finite difference method to nodes 1 and 2.

(*a*) Node 1 is an interior node, and the *explicit* finite difference formulation at that node is obtained directly from Eq. 5–47 by setting $m = 1$:

$$T_1^{i+1} = \tau(T_0 + T_2^i) + (1 - 2\tau)\,T_1^i + \tau\frac{\dot g_1 \Delta x^2}{k} \qquad (1)$$

Node 2 is a boundary node subjected to convection, and the finite difference formulation at that node is obtained by writing an energy balance on the volume element of thickness $\Delta x/2$ at that boundary by assuming heat transfer to be into the medium at all sides (Fig. 5–45):

$$hA(T_\infty - T_2^i) + kA\frac{T_1^i - T_2^i}{\Delta x} + \dot g_2 A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2}C\frac{T_2^{i+1} - T_2^i}{\Delta x}$$

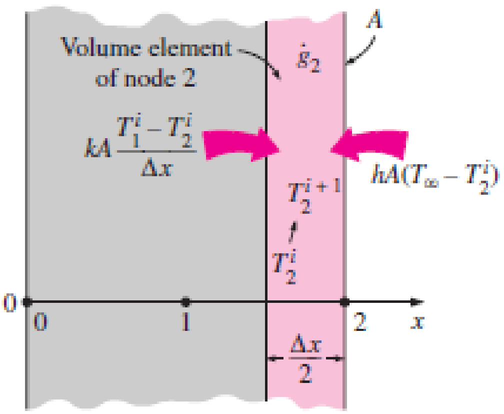


FIGURE 5–45
Schematic for the explicit finite difference formulation of the convection condition at the right boundary of a plane wall.

Dividing by $kA/2\Delta x$ and using the definitions of thermal diffusivity $\alpha = k/\rho C$ and the dimensionless mesh Fourier number $\tau = \alpha\Delta t/(\Delta x)^2$ gives

$$\frac{2h\Delta x}{k}(T_\infty - T_2^i) + 2(T_1^i - T_2^i) + \frac{\dot{g}_2\Delta x^2}{k} = \frac{T_2^{i+1} - T_2^i}{\tau}$$

which can be solved for T_2^{i+1} to give

$$T_2^{i+1} = \left(1 - 2\tau - 2\tau\frac{h\Delta x}{k}\right)T_2^i + \tau\left(2T_1^i + 2\frac{h\Delta x}{k}T_\infty + \frac{\dot{g}_2\Delta x^2}{k}\right) \quad \textbf{(2)}$$

Note that we did not use the superscript i for quantities that do not change with time. Next we need to determine the upper limit of the time step Δt from the stability criterion, which requires the coefficient of T_1^i in Equation 1 and the coefficient of T_2^i in the second equation to be greater than or equal to zero. The coefficient of T_2^i is smaller in this case, and thus the stability criterion for this problem can be expressed as

$$1 - 2\tau - 2\tau\frac{h\Delta x}{k} \geq 0 \quad \rightarrow \quad \tau \leq \frac{1}{2(1 + h\Delta x/k)} \quad \rightarrow \quad \Delta t \leq \frac{\Delta x^2}{2\alpha(1 + h\Delta x/k)}$$

since $\tau = \alpha\Delta t/(\Delta x)^2$. Substituting the given quantities, the maximum allowable value of the time step is determined to be

$$\Delta t \leq \frac{(0.02\text{ m})^2}{2(12.5 \times 10^{-6}\text{ m}^2/\text{s})[1 + (45\text{ W/m}^2\cdot{}^\circ\text{C})(0.02\text{ m})/28\text{ W/m}\cdot{}^\circ\text{C}]} = 15.5\text{ s}$$

Therefore, any time step less than 15.5 s can be used to solve this problem. For convenience, let us choose the time step to be $\Delta t = 15$ s. Then the mesh Fourier number becomes

$$\tau = \frac{\alpha\Delta t}{(\Delta x)^2} = \frac{(12.5 \times 10^{-6}\text{ m}^2/\text{s})(15\text{ s})}{(0.02\text{ m})^2} = 0.46875 \quad (\text{for } \Delta t = 15\text{ s})$$

Substituting this value of τ and other given quantities, the explicit finite difference equations (1) and (2) developed here reduce to

$$T_1^{i+1} = 0.0625T_1^i + 0.46875T_2^i + 33.482$$
$$T_2^{i+1} = 0.9375T_1^i + 0.032366T_2^i + 34.386$$

The initial temperature of the medium at $t = 0$ and $i = 0$ is given to be 200°C throughout, and thus $T_1^0 = T_2^0 = 200$°C. Then the nodal temperatures at T_1^1 and T_2^1 at $t = \Delta t = 15$ s are determined from these equations to be

$$T_1^1 = 0.0625T_1^0 + 0.46875T_2^0 + 33.482$$
$$= 0.0625 \times 200 + 0.46875 \times 200 + 33.482 = 139.7°\text{C}$$
$$T_2^1 = 0.9375T_1^0 + 0.032366T_2^0 + 34.386$$
$$= 0.9375 \times 200 + 0.032366 \times 200 + 34.386 = 228.4°\text{C}$$

Similarly, the nodal temperatures T_1^2 and T_2^2 at $t = 2\Delta t = 2 \times 15 = 30$ s are determined to be

$$T_1^2 = 0.0625T_1^1 + 0.46875T_2^1 + 33.482$$
$$= 0.0625 \times 139.7 + 0.46875 \times 228.4 + 33.482 = 149.3°\text{C}$$
$$T_2^2 = 0.9375T_1^1 + 0.032366T_2^1 + 34.386$$
$$= 0.9375 \times 139.7 + 0.032366 \times 228.4 + 34.386 = 172.8°\text{C}$$

Continuing in the same manner, the temperatures at nodes 1 and 2 are determined for $i = 1, 2, 3, 4, 5, \ldots, 50$ and are given in Table 5–2. Therefore, the temperature at the exposed boundary surface 2.5 min after the start of cooling is

$$T_L^{2.5\text{ min}} = T_2^{10} = 139.0°\text{C}$$

TABLE 5–2

The variation of the nodal temperatures in Example 5–5 with time obtained by the *explicit* method

Time Step, i	Time, s	Node Temperature, °C: T_1^i	Node Temperature, °C: T_2^i
0	0	200.0	200.0
1	15	139.7	228.4
2	30	149.3	172.8
3	45	123.8	179.9
4	60	125.6	156.3
5	75	114.6	157.1
6	90	114.3	146.9
7	105	109.5	146.3
8	120	108.9	141.8
9	135	106.7	141.1
10	150	106.3	139.0
20	300	103.8	136.1
30	450	103.7	136.0
40	600	103.7	136.0

(*b*) Node 1 is an interior node, and the *implicit* finite difference formulation at that node is obtained directly from Eq. 5–49 by setting $m = 1$:

$$\tau T_0 - (1 + 2\tau)\, T_1^{i+1} + \tau T_2^{i+1} + \tau \frac{\dot{g}_0\, \Delta x^2}{k} + T_1^i = 0 \tag{3}$$

Node 2 is a boundary node subjected to convection, and the implicit finite difference formulation at that node can be obtained from this formulation by expressing the left side of the equation at time step $i + 1$ instead of i as

$$\frac{2h\Delta x}{k}(T_\infty - T_2^{i+1}) + 2(T_1^{i+1} - T_2^{i+1}) + \frac{\dot{g}_2\, \Delta x^2}{k} = \frac{T_2^{i+1} - T_2^i}{\tau}$$

which can be rearranged as

$$2\tau T_1^{i+1} - \left(1 + 2\tau + 2\tau \frac{h\Delta x}{k}\right) T_2^{i+1} + 2\tau \frac{h\Delta x}{k} T_\infty + \tau \frac{\dot{g}_2\, \Delta x^2}{k} + T_2^i = 0 \tag{4}$$

Again we did not use the superscript i or $i + 1$ for quantities that do not change with time. The implicit method imposes no limit on the time step, and thus we can choose any value we want. However, we will again choose $\Delta t = 15$ s, and thus $\tau = 0.46875$, to make a comparison with part (*a*) possible. Substituting this value of τ and other given quantities, the two implicit finite difference equations developed here reduce to

$$-1.9375 T_1^{i+1} + 0.46875 T_2^{i+1} + T_1^i + 33.482 = 0$$
$$0.9375 T_1^{i+1} - 1.9676 T_2^{i+1} + T_2^i + 34.386 = 0$$

Again $T_1^0 = T_2^0 = 200°C$ at $t = 0$ and $i = 0$ because of the initial condition, and for $i = 0$, these two equations reduce to

$$-1.9375 T_1^1 + 0.46875 T_2^1 + 200 + 33.482 = 0$$
$$0.9375 T_1^1 - 1.9676 T_2^1 + 200 + 34.386 = 0$$

The unknown nodal temperatures T_1^1 and T_2^1 at $t = \Delta t = 15$ s are determined by solving these two equations simultaneously to be

$$T_1^1 = 168.8°C \qquad \text{and} \qquad T_2^1 = 199.6°C$$

TABLE 5–3

The variation of the nodal temperatures in Example 5–5 with time obtained by the *implicit* method

Time Step, i	Time, s	Node Temperature, °C T_1^i	T_2^i
0	0	200.0	200.0
1	15	168.8	199.6
2	30	150.5	190.6
3	45	138.6	180.4
4	60	130.3	171.2
5	75	124.1	163.6
6	90	119.5	157.6
7	105	115.9	152.8
8	120	113.2	149.0
9	135	111.0	146.1
10	150	109.4	143.9
20	300	104.2	136.7
30	450	103.8	136.1
40	600	103.8	136.1

Similarly, for $i = 1$, these equations reduce to

$$-1.9375T_1^2 + 0.46875T_2^2 + 168.8 + 33.482 = 0$$
$$0.9375T_1^2 - 1.9676T_2^2 + 199.6 + 34.386 = 0$$

The unknown nodal temperatures T_1^2 and T_2^2 at $t = \Delta t = 2 \times 15 = 30$ s are determined by solving these two equations simultaneously to be

$$T_1^2 = 150.5°\text{C} \qquad \text{and} \qquad T_2^2 = 190.6°\text{C}$$

Continuing in this manner, the temperatures at nodes 1 and 2 are determined for $i = 2, 3, 4, 5, \ldots, 40$ and are listed in Table 5–3, and the temperature at the exposed boundary surface (node 2) 2.5 min after the start of cooling is obtained to be

$$T_L^{2.5\text{ min}} = T_2^{10} = 143.9°\text{C}$$

which is close to the result obtained by the explicit method. Note that either method could be used to obtain satisfactory results to transient problems, except, perhaps, for the first few time steps. The implicit method is preferred when it is desirable to use large time steps, and the explicit method is preferred when one wishes to avoid the simultaneous solution of a system of algebraic equations.

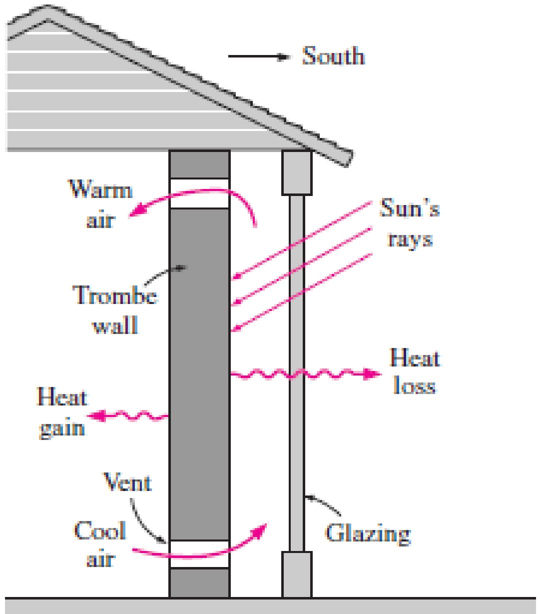


FIGURE 5–46
Schematic of a Trombe wall (Example 5–6).

TABLE 5–4
The hourly variation of monthly average ambient temperature and solar heat flux incident on a vertical surface for January in Reno, Nevada

Time of Day	Ambient Temperature, °F	Solar Radiation, Btu/h · ft²
7 AM–10 AM	33	114
10 AM–1 PM	43	242
1 PM–4 PM	45	178
4 PM–7 PM	37	0
7 PM–10 PM	32	0
10 PM–1 AM	27	0
1 AM–4 AM	26	0
4 AM–7 AM	25	0

EXAMPLE 5–6 Solar Energy Storage in Trombe Walls

Dark painted thick masonry walls called Trombe walls are commonly used on south sides of passive solar homes to absorb solar energy, store it during the day, and release it to the house during the night (Fig. 5–46). The idea was proposed by E. L. Morse of Massachusetts in 1881 and is named after Professor Felix Trombe of France, who used it extensively in his designs in the 1970s. Usually a single or double layer of glazing is placed outside the wall and transmits most of the solar energy while blocking heat losses from the exposed surface of the wall to the outside. Also, air vents are commonly installed at the bottom and top of the Trombe walls so that the house air enters the parallel flow channel between the Trombe wall and the glazing, rises as it is heated, and enters the room through the top vent.

Consider a house in Reno, Nevada, whose south wall consists of a 1-ft-thick Trombe wall whose thermal conductivity is $k = 0.40$ Btu/h · ft · °F and whose thermal diffusivity is $\alpha = 4.78 \times 10^{-6}$ ft²/s. The variation of the ambient temperature T_{out} and the solar heat flux $\dot{q}_{solar}$ incident on a south-facing vertical surface throughout the day for a typical day in January is given in Table 5–4 in 3-h intervals. The Trombe wall has single glazing with an absorptivity-transmissivity product of $\kappa = 0.77$ (that is, 77 percent of the solar energy incident is absorbed by the exposed surface of the Trombe wall), and the average combined heat transfer coefficient for heat loss from the Trombe wall to the ambient is determined to be $h_{out} = 0.7$ Btu/h · ft² · °F. The interior of the house is maintained at $T_{in} = 70$°F at all times, and the heat transfer coefficient at the interior surface of the Trombe wall is $h_{in} = 1.8$ Btu/h · ft² · °F. Also, the vents on the Trombe wall are kept closed, and thus the only heat transfer between the air in the house and the Trombe wall is through the interior surface of the wall. Assuming the temperature of the Trombe wall to vary linearly between 70°F at the interior surface and 30°F at the exterior surface at 7 AM and using the explicit finite difference method with a uniform nodal spacing of $\Delta x = 0.2$ ft, determine the temperature distribution along the thickness of the Trombe wall after 12, 24, 36, and 48 h. Also, determine the net amount of heat transferred to the house from the Trombe wall during the first day and the second day. Assume the wall is 10 ft high and 25 ft long.

SOLUTION The passive solar heating of a house through a Trombe wall is considered. The temperature distribution in the wall in 12-h intervals and the amount of heat transfer during the first and second days are to be determined.

Assumptions **1** Heat transfer is one-dimensional since the exposed surface of the wall is large relative to its thickness. **2** Thermal conductivity is constant. **3** The heat transfer coefficients are constant.

Properties The wall properties are given to be $k = 0.40$ Btu/h · ft · °F, $\alpha = 4.78 \times 10^{-6}$ ft²/s, and $\kappa = 0.77$.

Analysis The nodal spacing is given to be $\Delta x = 0.2$ ft, and thus the total number of nodes along the Trombe wall is

$$M = \frac{L}{\Delta x} + 1 = \frac{1 \text{ ft}}{0.2 \text{ ft}} + 1 = 6$$

We number the nodes as 0, 1, 2, 3, 4, and 5, with node 0 on the interior surface of the Trombe wall and node 5 on the exterior surface, as shown in Figure 5–47. Nodes 1 through 4 are interior nodes, and the explicit finite difference formulations of these nodes are obtained directly from Eq. 5–47 to be

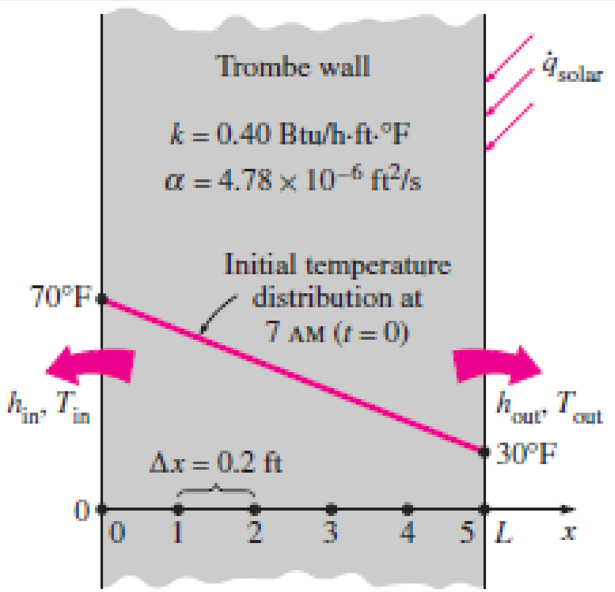


FIGURE 5–47
The nodal network for the Trombe wall discussed in Example 5–6.

$$\text{Node 1 } (m = 1): \quad T_1^{i+1} = \tau(T_0^i + T_2^i) + (1 - 2\tau)T_1^i \tag{1}$$

$$\text{Node 2 } (m = 2): \quad T_2^{i+1} = \tau(T_1^i + T_3^i) + (1 - 2\tau)T_2^i \tag{2}$$

$$\text{Node 3 } (m = 3): \quad T_3^{i+1} = \tau(T_2^i + T_4^i) + (1 - 2\tau)T_3^i \tag{3}$$

$$\text{Node 4 } (m = 4): \quad T_4^{i+1} = \tau(T_3^i + T_5^i) + (1 - 2\tau)T_4^i \tag{4}$$

The interior surface is subjected to convection, and thus the explicit formulation of node 0 can be obtained directly from Eq. 5–51 to be

$$T_0^{i+1} = \left(1 - 2\tau - 2\tau\frac{h_{\text{in}}\,\Delta x}{k}\right)T_0^i + 2\tau T_1^i + 2\tau\frac{h_{\text{in}}\,\Delta x}{k}T_{\text{in}}$$

Substituting the quantities h_{in}, Δx, k, and T_{in}, which do not change with time, into this equation gives

$$T_0^{i+1} = (1 - 3.80\tau)\,T_0^i + \tau(2T_1^i + 126.0) \tag{5}$$

The exterior surface of the Trombe wall is subjected to convection as well as to heat flux. The explicit finite difference formulation at that boundary is obtained by writing an energy balance on the volume element represented by node 5,

$$h_{\text{out}}A(T_{\text{out}}^i - T_5^i) + \kappa A\dot{q}_{\text{solar}}^i + kA\frac{T_4^i - T_5^i}{\Delta x} = \rho A\frac{\Delta x}{2}C\frac{T_5^{i+1} - T_5^i}{\Delta t} \quad \textbf{(5-53)}$$

which simplifies to

$$T_5^{i+1} = \left(1 - 2\tau - 2\tau\frac{h_{\text{out}}\Delta x}{k}\right)T_5^i + 2\tau T_4^i + 2\tau\frac{h_{\text{out}}\Delta x}{k}T_{\text{out}}^i + 2\tau\frac{\kappa\dot{q}_{\text{solar}}^i\Delta x}{k} \quad \textbf{(5-54)}$$

where $\tau = \alpha\Delta t/\Delta x^2$ is the dimensionless mesh Fourier number. Note that we kept the superscript i for quantities that vary with time. Substituting the quantities h_{out}, Δx, k, and κ, which do not change with time, into this equation gives

$$T_5^{i+1} = (1 - 2.70\tau)\,T_5^i + \tau(2T_4^i + 0.70T_{\text{out}}^i + 0.770\dot{q}_{\text{solar}}^i) \quad \textbf{(6)}$$

where the unit of $\dot{q}_{\text{solar}}^i$ is Btu/h · ft².

Next we need to determine the upper limit of the time step Δt from the stability criterion since we are using the explicit method. This requires the identification of the smallest primary coefficient in the system. We know that the boundary nodes are more restrictive than the interior nodes, and thus we examine the formulations of the boundary nodes 0 and 5 only. The smallest and thus the most restrictive primary coefficient in this case is the coefficient of T_0^i in the formulation of node 0 since $1 - 3.8\tau < 1 - 2.7\tau$, and thus the stability criterion for this problem can be expressed as

$$1 - 3.80\tau \geq 0 \quad \rightarrow \quad \tau = \frac{\alpha\Delta x}{\Delta x^2} \leq \frac{1}{3.80}$$

Substituting the given quantities, the maximum allowable value of the time step is determined to be

$$\Delta t \leq \frac{\Delta x^2}{3.80\alpha} = \frac{(0.2\text{ ft})^2}{3.80 \times (4.78 \times 10^{-6}\text{ ft}^2/\text{s})} = 2202\text{ s}$$

Therefore, any time step less than 2202 s can be used to solve this problem. For convenience, let us choose the time step to be $\Delta t = 900$ s = 15 min. Then the mesh Fourier number becomes

$$\tau = \frac{\alpha \Delta t}{(\Delta x)^2} = \frac{(4.78 \times 10^{-6} \text{ ft}^2/\text{s})(900 \text{ s})}{(0.2 \text{ ft})^2} = 0.10755 \qquad (\text{for } \Delta t = 15 \text{ min})$$

Initially (at 7 AM or $t = 0$), the temperature of the wall is said to vary linearly between 70°F at node 0 and 30°F at node 5. Noting that there are five nodal spacings of equal length, the temperature change between two neighboring nodes is (70 − 30)°F/5 = 8°F. Therefore, the initial nodal temperatures are

$$T_0^0 = 70°\text{F}, \qquad T_1^0 = 62°\text{F}, \qquad T_2^0 = 54°\text{F},$$
$$T_3^0 = 46°\text{F}, \qquad T_4^0 = 38°\text{F}, \qquad T_5^0 = 30°\text{F}$$

Then the nodal temperatures at $t = \Delta t = 15$ min (at 7:15 AM) are determined from these equations to be

$$\begin{aligned}
T_0^1 &= (1 - 3.80\tau)\, T_0^0 + \tau(2T_1^0 + 126.0) \\
&= (1 - 3.80 \times 0.10755)\, 70 + 0.10755(2 \times 62 + 126.0) = 68.3°\text{ F} \\
T_1^1 &= \tau(T_0^0 + T_2^0) + (1 - 2\tau)\, T_1^0 \\
&= 0.10755(70 + 54) + (1 - 2 \times 0.10755)62 = 62°\text{F} \\
T_2^1 &= \tau(T_1^0 + T_3^0) + (1 - 2\tau)\, T_2^0 \\
&= 0.10755(62 + 46) + (1 - 2 \times 0.10755)54 = 54°\text{F} \\
T_3^1 &= \tau(T_2^0 + T_4^0) + (1 - 2\tau)\, T_3^0 \\
&= 0.10755(54 + 38) + (1 - 2 \times 0.10755)46 = 46°\text{F} \\
T_4^1 &= \tau(T_3^0 + T_5^0) + (1 - 2\tau)\, T_4^0 \\
&= 0.10755(46 + 30) + (1 - 2 \times 0.10755)38 = 38°\text{F} \\
T_5^1 &= (1 - 2.70\tau)\, T_5^0 + \tau(2T_4^0 + 0.70T_{\text{out}}^0 + 0.770\dot{q}_{\text{solar}}^0) \\
&= (1 - 2.70 \times 0.10755)30 + 0.10755(2 \times 38 + 0.70 \times 33 + 0.770 \times 114) \\
&= 41.4°\text{F}
\end{aligned}$$

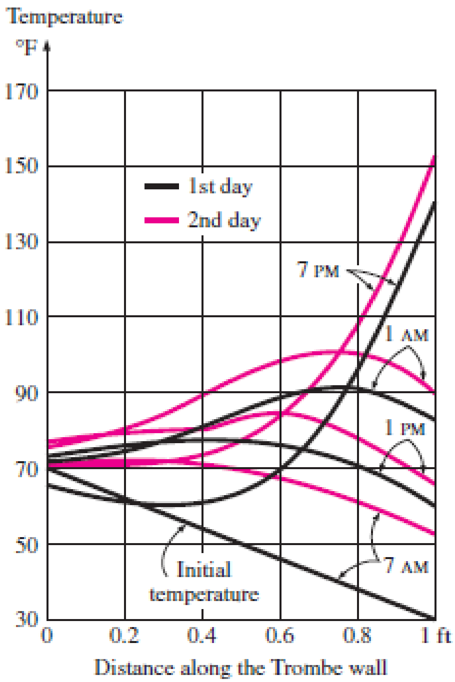


FIGURE 5–48
The variation of temperatures in the Trombe wall discussed in Example 5–6.

Note that the inner surface temperature of the Trombe wall dropped by 1.7°F and the outer surface temperature rose by 11.4°F during the first time step while the temperatures at the interior nodes remained the same. This is typical of transient problems in mediums that involve no heat generation. The nodal temperatures at the following time steps are determined similarly with the help of a computer. Note that the data for ambient temperature and the incident solar radiation change every 3 hours, which corresponds to 12 time steps, and this must be reflected in the computer program. For example, the value of $\dot{q}'_{solar}$ must be taken to be $\dot{q}'_{solar} = 75$ for $i = 1\text{–}12$, $\dot{q}'_{solar} = 242$ for $i = 13\text{–}24$, $\dot{q}'_{solar} = 178$ for $i = 25\text{–}36$, and $\dot{q}'_{solar} = 0$ for $i = 37\text{–}96$.

The results after 6, 12, 18, 24, 30, 36, 42, and 48 h are given in Table 5–5 and are plotted in Figure 5–48 for the first day. Note that the interior temperature of the Trombe wall drops in early morning hours, but then rises as the solar energy absorbed by the exterior surface diffuses through the wall. The exterior surface temperature of the Trombe wall rises from 30 to 142°F in just 6 h because of the solar energy absorbed, but then drops to 53°F by next morning as a result of heat loss at night. Therefore, it may be worthwhile to cover the outer surface at night to minimize the heat losses.

TABLE 5–5

The temperatures at the nodes of a Trombe wall at various times

Time	Time Step, i	Nodal Temperatures, °F					
		T_0	T_1	T_2	T_3	T_4	T_5
0 h (7 AM)	0	70.0	62.0	54.0	46.0	38.0	30.0
6 h (1 PM)	24	65.3	61.7	61.5	69.7	94.1	142.0
12 h (7 PM)	48	71.6	74.2	80.4	88.4	91.7	82.4
18 h (1 AM)	72	73.3	75.9	77.4	76.3	71.2	61.2
24 h (7 AM)	96	71.2	71.9	70.9	67.7	61.7	53.0
30 h (1 PM)	120	70.3	71.1	74.3	84.2	108.3	153.2
36 h (7 PM)	144	75.4	81.1	89.4	98.2	101.0	89.7
42 h (1 AM)	168	75.8	80.7	83.5	83.0	77.4	66.2
48 h (7 AM)	192	73.0	75.1	72.2	66.0	66.0	56.3

The rate of heat transfer from the Trombe wall to the interior of the house during each time step is determined from Newton's law using the average temperature at the inner surface of the wall (node 0) as

$$Q^i_{\text{Trombe wall}} = \dot{Q}^i_{\text{Trombe wall}}\,\Delta t = h_{\text{in}}\,A(T^i_0 - T_{\text{in}})\,\Delta t = h_{\text{in}}\,A[(T^i_0 + T^{i-1}_0)/2 - T_{\text{in}}]\Delta t$$

Therefore, the amount of heat transfer during the first time step ($i = 1$) or during the first 15-min period is

$$\begin{aligned} Q^1_{\text{Trombe wall}} &= h_{\text{in}}\,A[(T^1_0 + T^0_0)/2 - T_{\text{in}}]\,\Delta t \\ &= (1.8\ \text{Btu/h}\cdot\text{ft}^2\cdot{}^\circ\text{F})(10 \times 25\ \text{ft}^2)[(68.3 + 70)/2 - 70^\circ\text{F}](0.25\ \text{h}) \\ &= -95.6\ \text{Btu} \end{aligned}$$

The negative sign indicates that heat is transferred to the Trombe wall from the air in the house, which represents a heat loss. Then the total heat transfer during a specified time period is determined by adding the heat transfer amounts for each time step as

$$Q_{\text{Trombe wall}} = \sum_{i=1}^{I} \dot{Q}^i_{\text{Trombe wall}} = \sum_{i=1}^{I} h_{\text{in}}\,A[(T^i_0 + T^{i-1}_0)/2 - T_{\text{in}}]\,\Delta t \qquad \textbf{(5-55)}$$

where I is the total number of time intervals in the specified time period. In this case $I = 48$ for 12 h, 96 for 24 h, and so on. Following the approach described here using a computer, the amount of heat transfer between the Trombe wall and the interior of the house is determined to be

$$\begin{aligned} Q_{\text{Trombe wall}} &= -17,048\ \text{Btu after 12 h} &&(-17,078\ \text{Btu during the first 12 h}) \\ Q_{\text{Trombe wall}} &= -2483\ \text{Btu after 24 h} &&(14,565\ \text{Btu during the second 12 h}) \\ Q_{\text{Trombe wall}} &= 5610\ \text{Btu after 36 h} &&(8093\ \text{Btu during the third 12 h}) \\ Q_{\text{Trombe wall}} &= 34,400\ \text{Btu after 48 h} &&(28,790\ \text{Btu during the fourth 12 h}) \end{aligned}$$

Therefore, the house loses 2483 Btu through the Trombe wall the first day as a result of the low start-up temperature but delivers a total of 36,883 Btu of heat to the house the second day. It can be shown that the Trombe wall will deliver even more heat to the house during the third day since it will start the day at a higher average temperature.

Two-Dimensional Transient Heat Conduction

$$k\Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k\Delta y \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \dot{e}_{m,n}\,\Delta x \Delta y = \rho \Delta x \Delta y\, c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

$$(\Delta x = \Delta y = l)$$

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{\dot{e}_{m,n} l^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

$\alpha = k/\rho c_p$ $\quad$ $\tau = \alpha \Delta t / l^2$

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{e}_{\text{node}} l^2}{k} = \frac{T_{\text{node}}^{i+1} - T_{\text{node}}^i}{\tau}$$

$$T_{\text{left}}^i + T_{\text{top}}^i + T_{\text{right}}^i + T_{\text{bottom}}^i - 4T_{\text{node}}^i + \frac{\dot{e}_{\text{node}}^i l^2}{k} = \frac{T_{\text{node}}^{i+1} - T_{\text{node}}^i}{\tau}$$

$$T_{\text{node}}^{i+1} = \tau(T_{\text{left}}^i + T_{\text{top}}^i + T_{\text{right}}^i + T_{\text{bottom}}^i) + (1 - 4\tau)\, T_{\text{node}}^i + \tau \frac{\dot{e}_{\text{node}}^i l^2}{k}$$

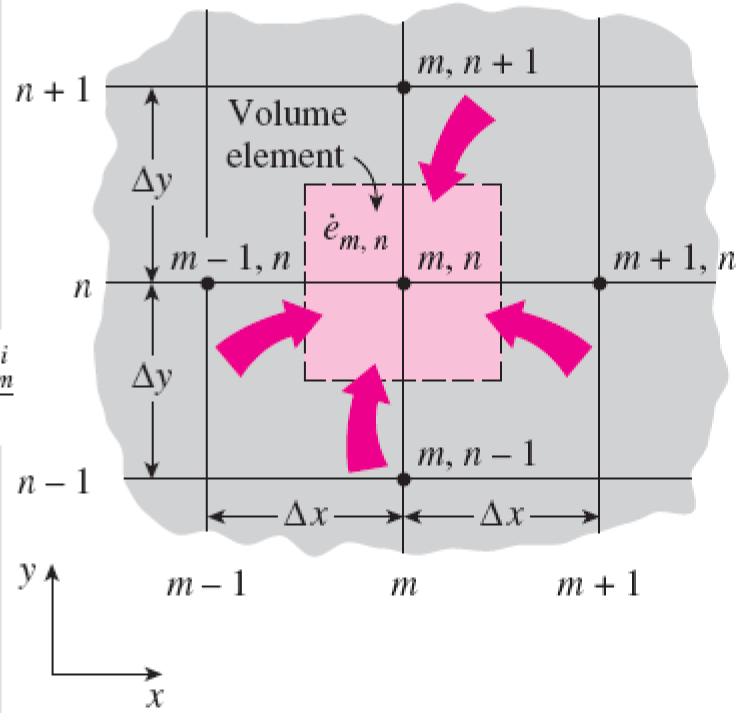


FIGURE 5–49

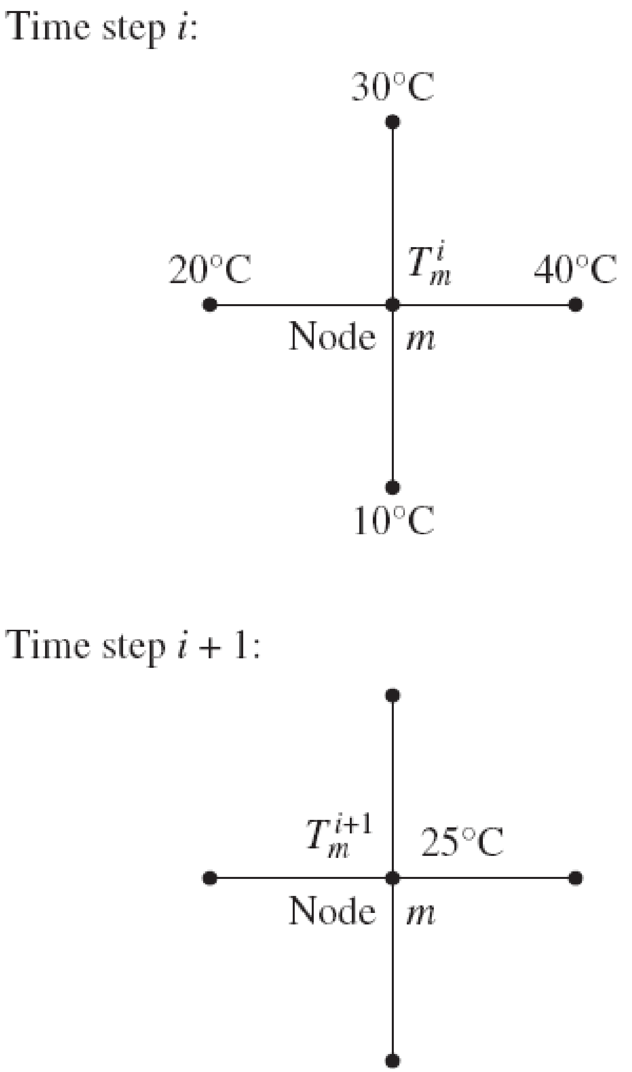
The volume element of a general interior node (m, n) for two-dimensional transient conduction in rectangular coordinates.

$$T_{\text{node}}^{i+1} = \tau(T_{\text{left}}^i + T_{\text{top}}^i + T_{\text{right}}^i + T_{\text{bottom}}^i) + (1 - 4\tau)\, T_{\text{node}}^i + \tau \frac{\dot{e}_{\text{node}}^i l^2}{k}$$

Explicit formulation

In the case of no heat generation and $\tau = 1/4$

$$T_{\text{node}}^{i+1} = (T_{\text{left}}^i + T_{\text{top}}^i + T_{\text{right}}^i + T_{\text{bottom}}^i)/4$$



Stability criterion

$$\tau = \frac{\alpha \Delta t}{l^2} \leq \frac{1}{4} \quad \text{(interior nodes, two-dimensional heat transfer in rectangular coordinates)}$$

FIGURE 5–50

In the case of no heat generation and $\tau = \frac{1}{4}$, the temperature of an interior node at the new time step is the average of the temperatures of its neighboring nodes at the previous time step.

EXAMPLE 5–7 Transient Two-Dimensional Heat Conduction in L-Bars

Consider two-dimensional transient heat transfer in an L-shaped solid body that is initially at a uniform temperature of 90°C and whose cross section is given in Figure 5–51. The thermal conductivity and diffusivity of the body are k = 15 W/m · °C and $\alpha = 3.2 \times 10^{-6}$ m²/s, respectively, and heat is generated in the body at a rate of $\dot{g} = 2 \times 10^6$ W/m³. The left surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of 90°C at all times. At time $t = 0$, the entire top surface is subjected to convection to ambient air at T_∞ = 25°C with a convection coefficient of h = 80 W/m² · °C, and the right surface is subjected to heat flux at a uniform rate of $\dot{q}_R$ = 5000 W/m². The nodal network of the problem consists of 15 equally spaced nodes with $\Delta x = \Delta y$ = 1.2 cm, as shown in the figure. Five of the nodes are at the bottom surface, and thus their temperatures are known. Using the explicit method, determine the temperature at the top corner (node 3) of the body after 1, 3, 5, 10, and 60 min.

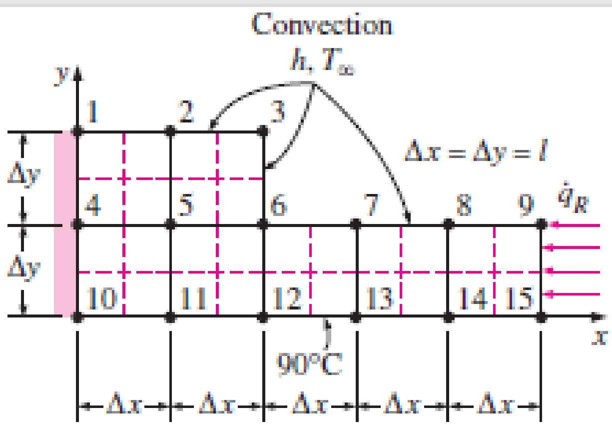


FIGURE 5–51
Schematic and nodal network for Example 5–7.

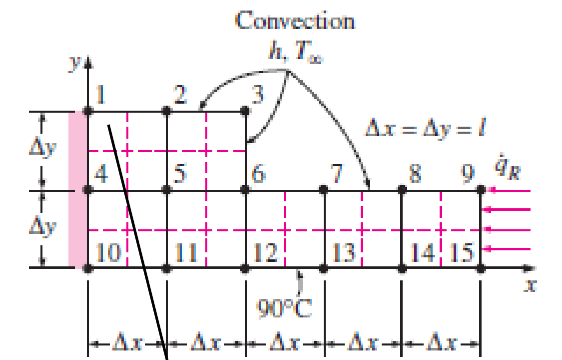
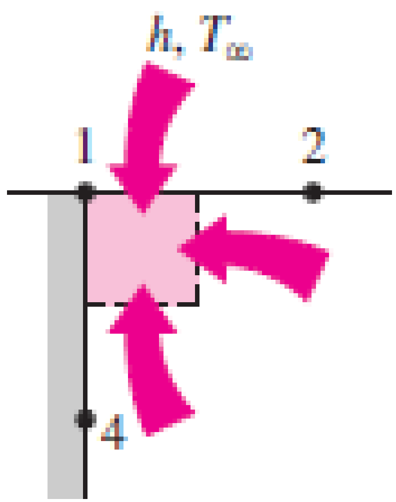


FIGURE 5–51



(a) Node 1 FIGURE 5–52

SOLUTION This is a transient two-dimensional heat transfer problem in rectangular coordinates, and it was solved in Example 5–3 for the steady case. Therefore, the solution of this transient problem should approach the solution for the steady case when the time is sufficiently large. The thermal conductivity and heat generation rate are given to be constants. We observe that all nodes are boundary nodes except node 5, which is an interior node. Therefore, we will have to rely on energy balances to obtain the finite difference equations. The region is partitioned among the nodes equitably as shown in the figure, and the explicit finite difference equations are determined on the basis of the energy balance for the transient case expressed as

$$\sum_{\text{All sides}} \dot{Q}^i + \dot{G}^i_{\text{element}} = \rho V_{\text{element}}\, C\, \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

The quantities h, T_∞, $\dot{g}$, and $\dot{q}_R$ do not change with time, and thus we do not need to use the superscript i for them. Also, the energy balance expressions are simplified using the definitions of thermal diffusivity $\alpha = k/\rho C$ and the dimensionless mesh Fourier number $\tau = \alpha\Delta t/l^2$, where $\Delta x = \Delta y = l$.

(*a*) Node 1. (Boundary node subjected to convection and insulation, Fig. 5–52*a*)

$$h\frac{\Delta x}{2}(T_\infty - T_1^i) + k\frac{\Delta y}{2}\frac{T_2^i - T_1^i}{\Delta x} + k\frac{\Delta x}{2}\frac{T_4^i - T_1^i}{\Delta y}$$
$$+ \dot{g}_1\frac{\Delta x}{2}\frac{\Delta y}{2} = \rho\frac{\Delta x}{2}\frac{\Delta y}{2}C\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Dividing by $k/4$ and simplifying,

$$\frac{2hl}{k}(T_\infty - T_1^i) + 2(T_2^i - T_1^i) + 2(T_4^i - T_1^i) + \frac{\dot{g}_1 l^2}{k} = \frac{T_1^{i+1} - T_1^i}{\tau}$$

which can be solved for T_1^{i+1} to give

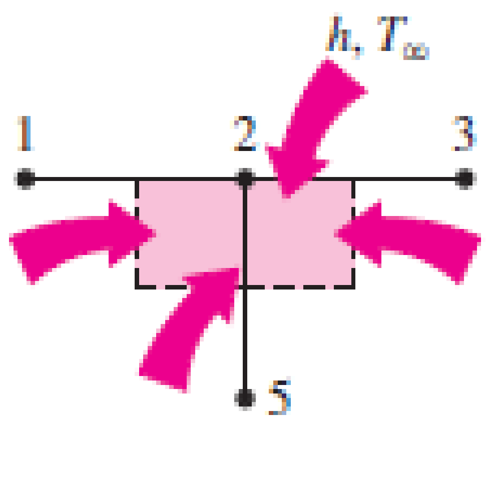
$$T_1^{i+1} = \left(1 - 4\tau - 2\tau\frac{hl}{k}\right)T_1^i + 2\tau\left(T_2^i + T_4^i + \frac{hl}{k}T_\infty + \frac{\dot{g}_1 l^2}{2k}\right)$$

(*b*) Node 2. (Boundary node subjected to convection, Fig. 5–52*b*)

$$h\Delta x(T_\infty - T_2^i) + k\frac{\Delta y}{2}\frac{T_3^i - T_2^i}{\Delta x} + k\Delta x\frac{T_5^i - T_2^i}{\Delta y}$$
$$+ k\frac{\Delta y}{2}\frac{T_1^i - T_2^i}{\Delta x} + \dot{g}_2\,\Delta x\frac{\Delta y}{2} = \rho\Delta x\frac{\Delta y}{2}C\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

Dividing by $k/2$, simplifying, and solving for T_2^{i+1} gives

$$T_2^{i+1} = \left(1 - 4\tau - 2\tau\frac{hl}{k}\right)T_2^i + \tau\left(T_1^i + T_3^i + 2T_5^i + \frac{2hl}{k}T_\infty + \frac{\dot{g}_2 l^2}{k}\right)$$



(*b*) Node 2 FIGURE 5–52

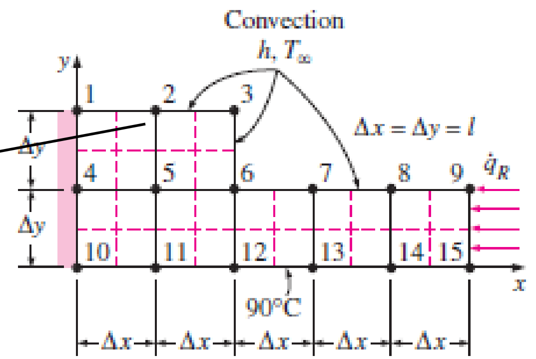


FIGURE 5–51

(*c*) Node 3. (Boundary node subjected to convection on two sides, Fig. 5–53*a*)

$$h\left(\frac{\Delta x}{2}+\frac{\Delta y}{2}\right)(T_\infty - T_3^i) + k\frac{\Delta x}{2}\frac{T_6^i - T_3^i}{\Delta y}$$

$$+\, k\frac{\Delta y}{2}\frac{T_2^i - T_3^i}{\Delta x} + \dot{g}_3\frac{\Delta x}{2}\frac{\Delta y}{2} = \rho\frac{\Delta x}{2}\frac{\Delta y}{2}\frac{T_3^{i+1} - T_3^i}{\Delta t}$$

Dividing by $k/4$, simplifying, and solving for T_3^{i+1} gives

$$T_3^{i+1} = \left(1 - 4\tau - 4\tau\frac{hl}{k}\right)T_3^i + 2\tau\left(T_4^i + T_6^i + 2\frac{hl}{k}T_\infty + \frac{\dot{g}_3 l^2}{2k}\right)$$

(*d*) Node 4. (On the insulated boundary, and can be treated as an interior node, Fig. 5–53*b*). Noting that $T_{10} = 90°C$, Eq. 5–60 gives

$$T_4^{i+1} = (1 - 4\tau)\,T_4^i + \tau\left(T_1^i + 2T_5^i + 90 + \frac{\dot{g}_4 l^2}{k}\right)$$

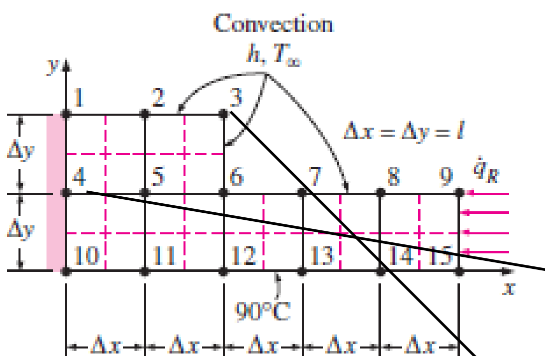
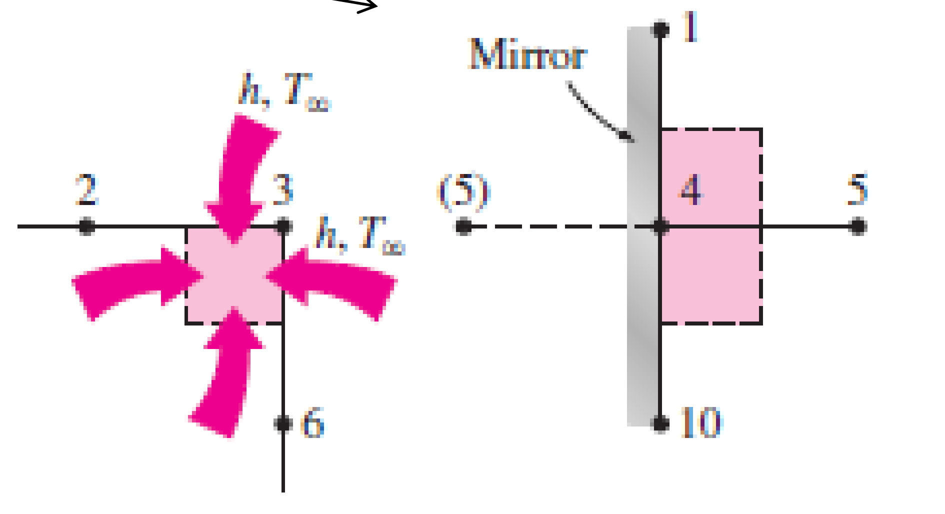


FIGURE 5–51



(*a*) Node 3

(*b*) Node 4

FIGURE 5–53

(*e*) Node 5. (Interior node, Fig. 5–54*a*). Noting that T_{11} = 90°C, Eq. 5–60 gives

$$T_5^{i+1} = (1 - 4\tau)\, T_5^i + \tau\left(T_2^i + T_4^i + T_6^i + 90 + \frac{\dot g_5 l^2}{k}\right)$$

(*f*) Node 6. (Boundary node subjected to convection on two sides, Fig. 5–54*b*)

$$h\left(\frac{\Delta x}{2} + \frac{\Delta y}{2}\right)(T_\infty - T_6^i) + k\frac{\Delta y}{2}\frac{T_7^i - T_6^i}{\Delta x} + k\Delta x\frac{T_{12}^i - T_6^i}{\Delta y} + k\Delta y\frac{T_5^i - T_6^i}{\Delta x}$$
$$+ \frac{\Delta x}{2}\frac{T_3^i - T_6^i}{\Delta y} + \dot g_6\frac{3\Delta x\Delta y}{4} = \rho\frac{3\Delta x\Delta y}{4}C\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

Dividing by $3k/4$, simplifying, and solving for T_6^{i+1} gives

$$T_6^{i+1} = \left(1 - 4\tau - 4\tau\frac{hl}{3k}\right)T_3^i$$
$$+ \frac{\tau}{3}\left[2T_3^i + 4T_5^i + 2T_7^i + 4 \times 90 + 4\frac{hl}{k}T_\infty + 3\frac{\dot g_6 l^2}{k}\right]$$

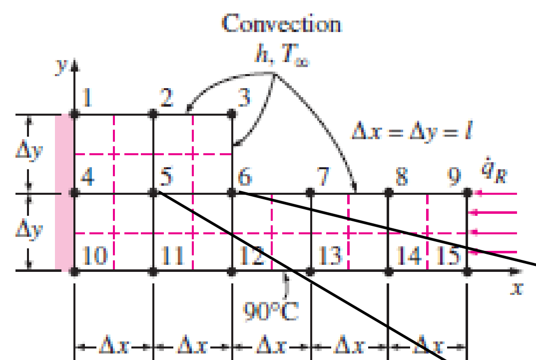
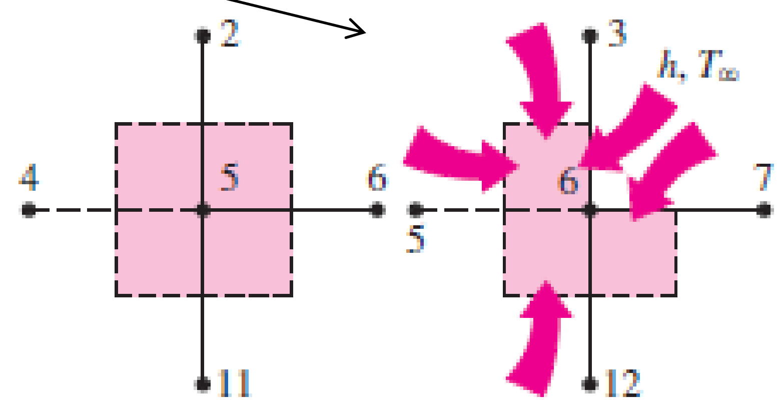


FIGURE 5–51



(*a*) Node 5 (*b*) Node 6

FIGURE 5–54

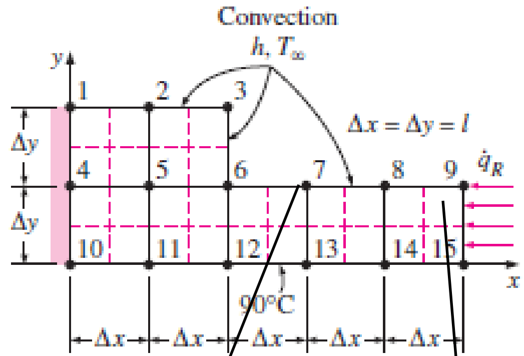


FIGURE 5–51

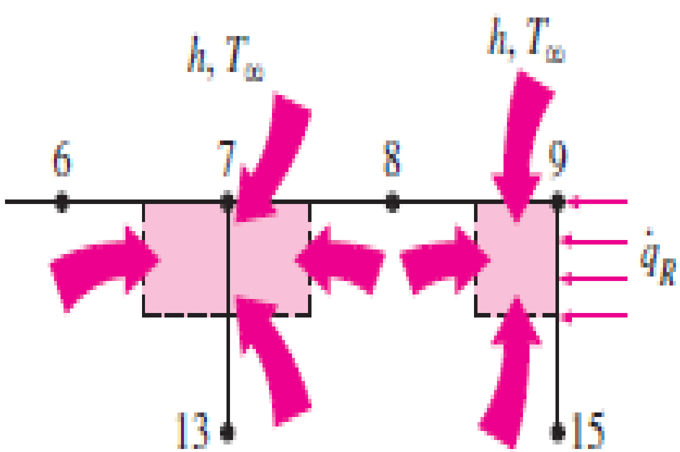


FIGURE 5–55

(*g*) Node 7. (Boundary node subjected to convection, Fig. 5–55)

$$h\Delta x(T_\infty - T_7^i) + k\frac{\Delta y}{2}\frac{T_8^i - T_7^i}{\Delta x} + k\Delta x\frac{T_{13}^i - T_7^i}{\Delta y} + k\frac{\Delta y}{2}\frac{T_6^i - T_7^i}{\Delta x} + \dot{g}_7\Delta x\frac{\Delta y}{2} = \rho\Delta x\frac{\Delta y}{2}C\frac{T_7^{i+1} - T_7^i}{\Delta t}$$

Dividing by $k/2$, simplifying, and solving for T_7^{i+1} gives

$$T_7^{i+1} = \left(1 - 4\tau - 2\tau\frac{hl}{k}\right)T_7^i + \tau\left[T_6^i + T_8^i + 2\times 90 + \frac{2hl}{k}T_\infty + \frac{\dot{g}_7 l^2}{k}\right]$$

(*h*) Node 8. This node is identical to node 7, and the finite difference formulation of this node can be obtained from that of node 7 by shifting the node numbers by 1 (i.e., replacing subscript *m* by subscript *m* + 1). It gives

$$T_8^{i+1} = \left(1 - 4\tau - 2\tau\frac{hl}{k}\right)T_8^i + \tau\left[T_7^i + T_9^i + 2\times 90 + \frac{2hl}{k}T_\infty + \frac{\dot{g}_8 l^2}{k}\right]$$

(*i*) Node 9. (Boundary node subjected to convection on two sides, Fig. 5–55)

$$h\frac{\Delta x}{2}(T_\infty - T_9^i) + \dot{q}_R\frac{\Delta y}{2} k\frac{\Delta x}{2}\frac{T_{15}^i - T_9^i}{\Delta y} + \frac{k\Delta y}{2}\frac{T_8^i - T_9^i}{\Delta x} + \dot{g}_9\frac{\Delta x}{2}\frac{\Delta y}{2} = \rho\frac{\Delta x}{2}\frac{\Delta y}{2}C\frac{T_9^{i+1} - T_9^i}{\Delta t}$$

Dividing by $k/4$, simplifying, and solving for T_9^{i+1} gives

$$T_9^{i+1} = \left(1 - 4\tau - 2\tau\frac{hl}{k}\right)T_9^i + 2\tau\left(T_8^i + 90 + \frac{\dot{q}_R l}{k} + \frac{hl}{k}T_\infty + \frac{\dot{g}_9 l^2}{2k}\right)$$

This completes the finite difference formulation of the problem. Next we need to determine the upper limit of the time step Δt from the stability criterion, which requires the coefficient of T_m^i in the T_m^{i+1} expression (the primary coefficient) to be greater than or equal to zero for all nodes. The smallest primary coefficient in the nine equations here is the coefficient of T_3^i in the expression, and thus the stability criterion for this problem can be expressed as

$$1 - 4\tau - 4\tau\frac{hl}{k} \geq 0 \quad \rightarrow \quad \tau \leq \frac{1}{4(1 + hl/k)} \quad \rightarrow \quad \Delta t \leq \frac{l^2}{4\alpha(1 + hl/k)}$$

since $\tau = \alpha\Delta t/l^2$. Substituting the given quantities, the maximum allowable value of the time step is determined to be

$$\Delta t \leq \frac{(0.012\text{ m})^2}{4(3.2 \times 10^{-6}\text{ m}^2/\text{s})[1 + (80\text{ W/m}^2\cdot{}^\circ\text{C})(0.012\text{ m})/(15\text{ W/m}\cdot{}^\circ\text{C})]} = 10.6\text{ s}$$

Therefore, any time step less than 10.6 s can be used to solve this problem. For convenience, let us choose the time step to be $\Delta t = 10$ s. Then the mesh Fourier number becomes

$$\tau = \frac{\alpha\Delta t}{l^2} = \frac{(3.2 \times 10^{-6}\text{ m}^2/\text{s})(10\text{ s})}{(0.012\text{ m})^2} = 0.222 \qquad (\text{for } \Delta t = 10\text{ s})$$

Substituting this value of τ and other given quantities, the developed transient finite difference equations simplify to

$$T_1^{i+1} = 0.0836T_1^i + 0.444(T_2^i + T_4^i + 11.2)$$
$$T_2^{i+1} = 0.0836T_2^i + 0.222(T_1^i + T_3^i + 2T_5^i + 22.4)$$
$$T_3^{i+1} = 0.0552T_3^i + 0.444(T_2^i + T_6^i + 12.8)$$
$$T_4^{i+1} = 0.112T_4^i + 0.222(T_1^i + 2T_5^i + 109.2)$$
$$T_5^{i+1} = 0.112T_1^i + 0.222(T_2^i + T_4^i + T_6^i + 109.2)$$

$$T_6^{i+1} = 0.0931T_6^i + 0.074(2T_3^i + 4T_5^i + 2T_7^i + 424)$$
$$T_7^{i+1} = 0.0836T_7^i + 0.222(T_6^i + T_8^i + 202.4)$$
$$T_8^{i+1} = 0.0836T_8^i + 0.222(T_7^i + T_9^i + 202.4)$$
$$T_9^{i+1} = 0.0836T_9^i + 0.444(T_8^i + 105.2)$$

Using the specified initial condition as the solution at time $t = 0$ (for $i = 0$), sweeping through these nine equations will give the solution at intervals of 10 s. The solution at the upper corner node (node 3) is determined to be 100.2, 105.9, 106.5, 106.6, and 106.6°C at 1, 3, 5, 10, and 60 min, respectively. Note that the last three solutions are practically identical to the solution for the steady case obtained in Example 5–3. This indicates that steady conditions are reached in the medium after about 5 min.

Interactive SS-T-CONDUCT Software

The **SS-T-CONDUCT** (Steady State and Transient Heat Conduction) software was developed by Ghajar and his co-workers and is available from the online learning center (www.mhhe.com/cengel) to the instructors and students.

The software is user-friendly and can be used to solve many of the one- and two-dimensional heat conduction problems with uniform energy generation in rectangular geometries discussed in this chapter.

For transient problems the explicit or the implicit solution method could be used.

The software has the following capabilities:

(a) Full and easy control of key numerical parameters (nodes and grids), material properties, and boundary conditions and parameters.
(b) The effect of parameter changes on the temperature distribution can be instantly viewed.
(c) The effect of stability criterion (Fourier number) for the explicit method can be explored.
(d) Several different ways of viewing the results on the screen or in print (output file):
   - Temperature results in a tabular form.
   - Temperature plots with time and distance for one-dimensional steady state and transient problems.
   - Shaded temperature plots for two-dimensional steady state problems.
   - Animation of shaded temperature plots for two-dimensional transient problems.
(e) A library of material properties (thermal conductivity and thermal diffusivity) built in the software. With this feature the effect of material property on the nodal temperatures can be explored.

The current version of the software has the following limitations:

(a) Rectangular geometries, expressed in Cartesian coordinates may be modeled.
(b) Uniform grid spacing.
(c) Boundary conditions for constant temperature, constant heat flux, and constant convection heat transfer coefficient.